SOME ASPECTS OF STOCHASTIC NONLINEAR PROGRAMMING PROBLEMS

A Thesis Submitted
In Partial Fulfilment of the Requirements
for the Degree of
DOCTOR OF PHILOSOPHY

By
Ram Shanker Sachan M.A

to the
Department of Mathematics
INDIAN INSTITUTE OF TECHNOLOGY
KANPUR

April – 1969

CERTIFICATE

This is to certify that the thesis entitled "Some Aspects of Stochastic Nonlinear Programming Problems" by Ram Shanker Sachan, for the award of the Degree of Doctor of Philosophy of the Indian Institute of Technology, Kanpur is a record of bonafide research work carried out by him under my supervision and guidance. The thesis has in my opinion reached the standard fulfilling the requirements for the Degree of Doctor of Philosophy. The results embodied in this thesis have not been submitted to any other university or institute for the award of any degree or diploma.

J N Kapur
J.N. KAPUR
Senior Professor of Mathematics

30th April - 1969.

ACKNOWLEDGEMENTS

I wish to express my sincere appreciation and great respect to Professor J.N. Kapur, M.A., Ph.D., F.A.Sc., F.N.A.Sc., F.N.I., F.I.M.A. (U.K.) under whose supervision this work is performed. He has always been a source of advice, patience and continuous encouragement of unestimable value for me throughout the past five years. It has been an unusual privilege to have worked under such a professor. No word can be adequate to express my gratefulness to him.

I am, also, grateful to Dr. S.K. Gupta, Dr. C.R. Bector, Shri R.P. Gupta, Shri Suresh, Shri Bhatt and especially to Shri N.K. Gogia, for their helpful discussions and valuable suggestions.

I, also, acknowledge gratefully the remarks and suggestions of the referee for the improvements in my paper [115].

Finally, my heartfelt thanks are also due to Shri S.K. Tewari for his fine typing of this thesis.

Ram Shanker Sachan

RAM SHANKER SACHAN

TABLE OF CONTENTS

LIST OF SYMBOLS

Ω, $\mathcal{B}$	: the sample spaces
$(\Omega, \mathcal{F}, P_r)$, $(\mathcal{B}, \mathcal{F}_1, \mu)$	: the probability spaces where $\mathcal{F}$ and $\mathcal{F}_1$ are σ-fields of the Borel subsets of Ω and $\mathcal{B}$ respectively and P_r and μ are the probability measures defined on $\mathcal{F}$ and $\mathcal{F}_1$ respectively. $\mathcal{F}$ and $\mathcal{F}_1$ are completed with respect to P_r and μ respectively.
ϕ	: the null set
E	: the expectation operator
σ_x	: the standard deviation of x
$\cup$, $\cap$	: the 'union' and 'intersection' respectively
$\subset$, $\supset$	: 'contained in' and 'contains' respectively
$\Rightarrow$, $\Leftarrow$ $\Leftrightarrow$	: 'implies', 'is implied by' 'implies and is implied by' respectively
$\in$	: belongs to the set
$\exists$	: there exists (exist)
$\forall$	: for all
E^n, R^n	: the Euclidean n-dimensional spaces
R^n_+	: the non-negative orthant of R^n
R	: the real line
$\langle \cdot , \cdot \rangle$	: the inner product of two elements
$\Vert \cdot \Vert$	: the Euclidean vector norm

∇ : the gradient operator

max., min. (sup., inf.): the maximum, minimum (supremum, infimum)

$\vee$ ($\wedge$) : sup. (inf.) of any two elements

P : the probability

Let ℓ, m and n be the positive integers

(m.n) : the expression number n, section m of the same chapter

(ℓ.m.n) : the expression, number n, section m of chapter ℓ .

$(m.n)_i$: the expression number n, section m of part i, where i = I or II.

SYNOPSIS

"Some Aspects of Stochastic Non-linear Programming Problems," a thesis submitted in partial fulfilment for the Degree of Doctor of Philosophy by Ram Shanker Sachan, M.A., to the Department of Mathematics, Indian Institute of Technology, Kanpur, April - 1969.

The present thesis consists of the theoretical discussion of some stochastic linear programming problems. It contains six chapters. The first chapter includes a discussion of stochastic programming problems and a brief survey of the research work done so far so as to place the contributions of the present thesis in their proper perspective. It also gives a summary of the present thesis.

Chapter II deals with a programming problem in which a nonlinear profit function is maximized subject to nonlinear constraints where all the parameters involved are random variables with known distributions. The problem is defined over an abstract vector space. The characterization of the problem is first obtained for the case where random variables are treated just as parameters. Later some important inequalities relating bounds for the programming problems are obtained for the stochastic version of the problem under the condition that the objective function is a convex function of its arguments. The inequalities obtained by Mangasarian "Nonlinear programming problems with stochastic objective functions" Man. Sc. Vol. 10, 1964 come out as particular cases of the above inequalities. Some bounds for the fat solutions of the problem are also obtained. It is further shown that these inequalities are satisfied for the case where

requirements are convex functions of costs or the costs are convex functions of the requirements. The inequalities are also deduced for the problem in which the objective function is a concave function of the random parameters. Several numerical examples are given in support of the results obtained.

The problem considered in Chapter III, is a quadratic version of the problem considered by Madansky "Dual Variables in two stage linear programming under uncertainty" J. Math. Anal. Appl. Vol. 6, 1963 . The problem is reformulated in a manner similar to that of Madansky and differing from his formulation at one or two places only. The emphasis is given on finding the optimality criteria and proving the saddle point theorem and theduality theorem for this problem as well as for the corresponding second-stage problem. These results are established by two different techniques (i) with the help of fundamental (Hahn-Banach) seperation theorem and (ii) by the method corresponding closely to that of Madansky.

In Chapter IV, the following type of the stochastic programming problems are considered:

(a) Stochastic linear programming problems with linear losses

(b) Stochastic linear programming problems with quadratic losses

(c) Stochastic quadratic programming problems with linear losses

where all the models (a), (b) and (c) are subject to linear 'deterministic as well as probabilistic' constraints, and

(d) Stochastic linear programming problems with linear losses subject to a random quadratic constraint and linear "deterministic as well as probalistic" constraints.

All these problems consist of two stage formulations. It is assumed that the random variables are known by their distributions. Some necessary and sufficient conditions for the existence of finite optima for the above problems are obtained. The decision equivalent deterministic programs obtained for the above problems come out to be convex with non-differentiable objective functions. It is remarked there that solving these deterministic programs is very difficult, but they can be solved via duality theorem. The dual program comes out to be a concave quadratic program. Some duality and optimality relations are established for the deterministic programs.

The two stage nonlinear programming problems under risk and uncertainty are considered in Chapter V, which is divided into two parts. In part one, the problem considered is that of two-stage minimization problem where the random variables are known by their distributions. The functions considered in the objective function are convex and in the constraints concave. The set of feasible solutions obtained is shown to be convex. The decision equivalent deterministic program which comes out to be convex, is obtained. The saddle point theorem and the optimality conditions are established for the second-stage program and then, for the two-stage program. It is remarked there that the above results will also hold for the nonconvex programming problems.

In the second part, the problem considered is similar to that considered in the first part, but there are additional constraints. Here, it is also assumed that the distributions of the random variables are known. The set of feasible solutions and the decision equivalent deterministic programs for this problem come out to be convex. The objective

function of the deterministic program is non-differentiable. Under some realistic assumptions the optimality and duality relations for the deterministic program are established.

Chapter VI is also divided into two parts. It considers the general symmetric dual programs with standard errors in the objective functions. The first part is devoted to establishing the symmetric duality and self duality for the quadratic type of non-differentiable programming problems. In the second part the symmetric duality and self-duality is developed for the general nonlinear programming problems with non-differentiable objective functions.

The work included in the thesis is based on the following research papers:

1. "On the nonlinear Stochastic Programming Problems" Cah. Cent. Etud. Rech. Oper. Vol. 10, No. 2, (00. 84-99) 1968.

2. "Two-stage Nonlinear Programming Under Uncertainty" Sent for publication in revised version.

3. "Duality in Quadratic Programming Problems Under Uncertainty" Sent for publication in revised version.

4. "Stochastic Programming Problems Under Risk and Uncertainty" to appear in Cah. Cent. Etud. Rech. Opr. Vol. 12, 1970.

5. "On the Nonlinear Programming Problems Under Risk and Uncertainty" to appear in the Bulletin of the Calcutta Branch of the Operational Research Society of India.

6. "Symmetric and Self-dual Programs with standard Errors in the Objective Functions" Sent for publication.

7. General 'Symmetric and Self-dual Programs with Standard Errors in the Objective Functions.

CHAPTER - I

INTRODUCTION

1. STOCHASTIC PROGRAMMING

Mathematical programming problems consist of maximising or minimising a given objective function subject to some constraints over the variables appearing in the objective functions. The variables of these problems are known as decision or control variables. The decision maker's object is to find out the values of these variables satisfying the restrictions imposed upon them such that the objective function is optimised. In all situations, the values of these variables will highly depend upon the parameters involved in the problem. If some or all of the parameters are changed by small quantities, the values of the variables and of the objective function will change accordingly. If such a situation arises i.e. if some of the parameters are subject to some variations, then the available methods such

as simplex, multiplex etc. fail to solve such problems. These types of problems where parameters are also variable are called the parametric programming problems. If the fluctuations of these parameters are random following some probability distributions, the problems are termed as the stochastic programming problems.

In stochastic programming problems, decisions are made to meet the future situations. In many of the real world problems arising in economics, commerce, industry, management, military science, social sciences etc. the stochastic programming models are finding rapidly increasing applications.

The purpose of this chapter is to give a brief account of stochastic programming together with a survey of these problems which are of special relevance for the present thesis. A short summary of the present thesis is also given.

There are two possible situations for the random variables appearing in the problem, namely (i) the distributions of the random variables are known completely and (ii) the random variables are not known completely by their distributions, but they can be estimated from the given data. These situations are respectively referred to as 'risk' and 'uncertainty' in the problem.

In most of the stochastic programming problems, the observations are carried out on the random variables concerned either before or after the decision has been taken. If they are observed before the optimal decisions are taken, the difficulty of solving the problem is reduced considerably. If the observations are performed on the random variables after the decisions have been taken, it is sure to incur some 'inaccuracies'

corresponding to the observed values and previous decision resulting in a loss, because the constraints may be violated some times. Such programs are called as the 'discrepancy cost' programs. In the programming problems, if constraints are satisfied at some prescribed probability level, they are known as chance-constraints, and the problem having these constraints is called the chance-constrained programming problem. If the distributions of the objective function is required to be found out, the problem is called the distributional problem.

There are several formulations of the stochastic programming problems which depend upon the nature of the random variables and the restrictions of the decision maker. The following are the four broad classes of stochastic programming problems: (i) Sequential (ii) Nonsequential (iii) Linear and (iv) Nonlinear. The later two classes can be embedded into the two former ones.

Sequential stochastic programs are also referred to as the multi-stage stochastic programs. In these problems, decisions are made at different stages such that the later decisions are functions of the previous decisions as well as the values of the random variables observed before the later decisions are taken. From the applicational point of view, these problems are more important than the non-sequential ones. Non-sequential stochastic programs are those in which only single decision is taken, and if several decisions are also made, then the later decisions may be taken independently of the random variables observed. Ordinary linear programming problem with parameters as random variables is known as the stochastic linear programming problem. Similarly, the non-linear stochastic programming problem can be understood.

There are several ways to reduce the effect of uncertainty from the problem. Some of these are: (i) to replace the random variable by its expected value, (ii) to replace the random variable by its maximum or minimum value or any pessimistic estimate of it (iii) to rephrase the problem into two-stage (multi-stage) problem where the decisions are taken compensating the previous stage 'inaccuracies' (iv) to recast the problem in terms of the preference functionals or in terms of the utility functions (v) to replace the random objective function by its cumulative distribution function or to use any criterion discussed in the following section.

2. DECISION CRITERIA IN STOCHASTIC PROGRAMMING

It is intended to give a precise interpretation to several decision criteria used in determining the admissible solution to the programming problems of the type

$$\underset{x \in X}{\text{maximise}} \quad f(x,b)$$

where $X = \{x \,/\, g(x,b) \geqslant 0,\ b \in \mathcal{B}\}$, b is a random vector with known distribution over the sample space $\mathcal{B} \subset E^N$. The decision criteria belong to two groups (a) criteria involving no probabilities i.e. interms of lower and upper bounds of the outcomes if they exist and (b) criteria involving probabilities i.e. in terms of the distributions of the outcomes.

The following are the criteria belonging to group (a).

(a-i) Maximin (also called minimax) pay-off criterion:

$$\underset{x \in X}{\text{Max}} \left[\underset{b \in \mathcal{B}}{\text{Inf}} \quad f(x,b) \right]$$

The, so called maximax criterion is

$$\max_{x \in X} \left[\sup_{b \in \mathcal{B}} f(x,b) \right]$$

(a-ii) The α- criterion:

$$\max_{x \in X} \left\{ \alpha \left[\sup_{b \in \mathcal{B}} f(x,b) \right] + (1 - \alpha) \left[\operatorname{Inf}_{b \in \mathcal{B}} f(x,b) \right] \right\}$$

where $0 \leq \alpha \leq 1$.

(a-iii) The minimax regret criterion:

$$\max_{x \in X} \left[\operatorname{Inf}_{b \in \mathcal{B}} h(x,b) \right]$$

where $h(x,b) = f(x,b) - \max_{x \in X} f(x,b)$ is a regret function.

(a-iv) Estimate criterion:

$$\max_{x \in X} f(x,\hat{b})$$

where $\hat{b}$ is any point (estimated) in $\mathcal{B}$.

(a-v) The principle of insufficient reason: If $\mathcal{B}$ consists of finite number (k) of elements each denoted by b_i , then this criterion states the following

$$\max_{x \in X} \left[\frac{1}{k} \sum_{i=1}^{k} f(x,b_i) \right]$$

The following criteria belong to the group (b).

(b-i) Maximum expected pay-off criterion:

$$\max_{x \in X} E\left[f(x,b) \right] = \int_{\mathcal{B}} f(x,b) \, d\mu(b)$$

where μ is the joint distribution function of b.

(b-ii) E - V criterion: This criterion tells that only efficient decisions should be considered. A decision vector $\bar{x} \in X$ is said to be efficient if it satisfies

$V\ f(\bar{x},b) \leq V\ f(x,b)$ for all $x \in X$ such that

$E\ f(x,b) \geqslant E\ f(\bar{x},b)$ and

$E\ f(\bar{x},b) \geqslant E\ f(x,b)$ for all $x \in X$ such that

$V\ f(x,b) \leq V\ f(\bar{x},b)$,

where E denotes the expectation and V the variance.

(b-iii) The truncated minimax criterion: If $f(x,b)$ is normally distributed for all $x \in X$, the criterion finds

$$\max_{x \in X} \phi f(x,b)$$

where $\phi f(x,b) = E\ f(x,b) + m\ \sigma f(x,b)$, $m \in R$ (real line) is a risk preference functional with a confidence limit interpretation and σ_f denotes the standard deviation of f.

(b-iv) α-fractile criterion:

$$\max_{x \in X} F_{\alpha} \left[f(x,b) \right]$$

where $0 < \alpha < 1$ is a predetermined constant and F_{α} is a α-fractile of $f(x,b)$. The α-fractile of the cumulative distribution function $F(y) = P\left[Y < y\right]$ is defined as $\sup\{y / F(y) \leq \alpha\}$.

(b-v) Aspiration criterion:

$$\max_{x \in X} P_k \left[f(x,b) \right]$$

where $P_k \left[f(x,b) \right]$ is the probability that $f(x,b)$ equals or exceeds the predetermined aspiration level k of the pay-off.

We now proceed to the next section in which we give a survey of research on stochastic programming problems.

3. A BRIEF SURVEY OF STOCHASTIC PROGRAMMING RESEARCH

Stochastic programming problems were first considered in 1955 independently by Dantzig [34] , Beale [3] and Tintner [124] . Dantzig and Beale considered similar types of problems of programming under uncertainty which is formulated in the two-stage program. Tintner considered the single-stage formulation of the problem in which the distribution of the optimum of the objective function was sought. Later on, some stochastic programming problems were considered by various authors by introducing the risk into the programming model.

In 1959, the chance-constrained programming was discussed by Charnes and Cooper [16] . From 1955 to 1962, some work was done in every aspect of stochastic programming. After 1962, the work in each area of stochastic programming was accelerated and much of the work (but not enough) has been done uptill now. In the present survey, three cases of the stochastic programming will be considered seperately. These are (i) programming under risk and uncertainty, (ii) chance-constrained programming and (iii) the distributional problems. The work of the present thesis is not included in this survey.

(3-1) Progr[illegible] Under Risk and Uncertain[illegible]

The two kinds of programming models (i) single-stage programming models and (ii) two-stage or multi-stage programming models, will be considered in this section. In the single-stage programming model, the entire probability distribution of the random variables appearing in the [illegible]roblem is known completely and no corrective actions are allowed to

compensate the risk incurred. Only, the minimum risk solutions are obtained.

The two-stage or multi-stage formulation of the stochastic programming model consists of the selective action under the imperfect knowledge of the random variables and the corrective actions under the perfect knowledge of the random variables. This can be interpreted as follows: The decision maker selects the activity levels for x and then observes the random variables. This is the first stage program. As soon as the observations are performed, he is finally allowed to take the corrective action y so as to compensate the inaccuracies which have occurred in the first stage program at the minimum cost. This is the second-stage program. The n-stage program can be defined in a similar way.

<u>The two-stage Program:</u>

A general n-stage program under uncertainty can be expressed in the following form

$$\text{Minimize:}\quad cx + E_{b_2}\left[c_2x_2 + E_{b_3}\left\{c_3x_3 + \text{———} + E_{b_n}(c_nx_n)\right\}\right]$$

subject to

$$\begin{aligned} &A_{11}x = b \\ &A_{21}x + A_{22}x_2 = b_2 \\ (3.1)\qquad &A_{32}x_2 + A_{33}x_3 = b_3 \\ &- - - - - - - - - \\ &A_{n,n-1}x_{n-1} + A_{nn}x_n = b_n \\ &x \geqslant 0,\ x_2 \geqslant 0,\ \text{———},\ x_n \geqslant 0\,, \end{aligned}$$

where c, c_2, c_3 —— c_n, b and A_{11}, A_{21} —— A_{nn} are constant vectors and constant matrices respectively b_2, b_3 —— b_n are randomvectors each element

of which is a random variable with known distribution and E_{b_i} denotes the expectation with respect to b_i, $i = 2, \text{-----} n$.

For $n = 2$, the program (3.1) reduces to the two-stage program

(3.2) $$\text{Min} \quad cx + E_{b_2}(c_2 x_2)$$

$$\text{subject to} \quad A_{11}x = b$$

$$A_{21}x + A_{22}x_2 = b_2$$

$$x, x_2 \geqslant 0$$

Dantzig [34] and Beale [3] were the first who considered this program. Dantzig [34] called this problem as 'here and now' solution problem. Tintner [125] proposed an 'active approach' solution for the similar type of the problem which can be easily reduced to 'here and now' solution problem as shown in [150].

If, instead of b_2, the vectors c, c_2 and matrices A_{21} & A_{22} are also random, the problem (3.2) is then called the stochastic programming problem with 'recourse' [95, 149, 150]. Under certain conditions imposed on A_{21} and A_{22}, the stochastic recourse problem reduces to the following:

'The relatively complete recourse': If $K_2 \supset K_1$, where K_1 and K_2 are as defined below. This will be the case when $b_2 - A_{21}x$ belongs to the convex cone generated by the columns of A_{22} with probability one for all $x \in K_1$.

'Complete recourse': If $K_2 = E^n$.

'Fixed recourse': If A_{22} is a constant matrix for all random $b_2 \in \Omega$, the sample space of b_2.

'Stable recourse': If A_{22} is fixed and equals to $[I, -I]$, where I is the identity matrix of appropriate order and $-I$ is its negative.

When, only the vector b_2 is random, the stable recourse problem is termed as the complete problem [95, 146, 148].

The Set of Feasible Solutions: In various papers [23,34,57,73,76,77,78,80] it is assumed that the set of feasible solutions exists for all random b_2 except for the sets of measure zero. This set of feasible solution is referred to as the 'permanently feasible' with probability one. Consider the following sets

$$K_1 = \{x \mid A_{11}x = b, \ x \geq 0\} \quad \text{and}$$

$$K_2 = \bigcap_{b_i \in \Omega} K_{2i} \quad , \ i = 2,3 \text{---} n$$

where $K_{2i} = \{x \mid A_{21}x = b_2 - A_{22}x_2, \text{ for some } x_2 \geq 0\}$.

The set K_1 is convex polyhedron, the set K_2 is a convex set. Then, in [36,85,95,139,146,147,148,149,150], it is shown that the set of feasible solutions $K = K_1 \cap K_2$ assumed to be not null, is convex. In [36, 147], it is also shown that the set K is not only convex but also polyhedral. The set K for the program (3.1) is also a polyhedron [147].

In case where the matrices A_{21} and A_{22} are also random, consider the second-stage program as defined in [149]

$$(3.3) \qquad F(x,w) = \min_{x_2 \geq 0} \ c_2(w)x_2$$

$$\text{subject to} \quad A_{22}(w)x_2 = b_2(w) - A_{21}(w)x.$$

Then, the following sets as defined in [149,150] the 'weak feasibility set' $K^2 = \{x \mid F(x,w) = +\infty \text{ with zero probability}\}$, the 'strong feasibility set' $K_1^2 = \{x \mid E\,F(x,w) < +\infty\}$ and the 'elementary feasibility set' $K^{2(w)} = \{x \mid F(x,w) < +\infty\}$ come out to be convex. For the fixed recourse, the set K^2 isclosed and convex. Under some restrictions the set K^2 is a convex polyhedron.

The Equivalent Deterministic Programs:

A programming problem: $\underset{x \in K}{\text{minimize}}\ f(x)$ is said to be the equivalent deterministic program for the problem (3.2) if sets of feasible solutions to both the problems are the same and optimal solutions for these problems coincide or become identical.

In order to find the optimal decision variables for the stochastic programming problems, it is customary to transform the stochastic programming problem into the equivalent deterministic programming problem which can be easily solved and gives the optimal decision for the original stochastic programming problem.

The nature of the equivalent deterministic programming problems depends upon the nature of the stochastic programming problems taken as parametric programming problems. Most of the problems considered in the literature are the stochastic linear programming problems whose equivalent deterministic models come out to be convex programs [3,4,23,28,34,35,36,42 to 46,73,85,135,136,138,146,148 to 150] . For the case of general convex programming under uncertainty, the equivalent deterministic forms are also convex [5,6,79,80,139] . The same holds for the multi-stage programming problems [34,85,146,147] .

There are several methods for finding the equivalent deterministic programs:

(i) Consider the second-stage program (3.3) for the problem (3.2). Then, by the theory of duality in linear programming one obtains

$$F(x,b_2) = Q(x,b_2) = \max \left\{ \pi'(b_2 - A_{21}x) \,/\, A'_{22}\pi \leq c_2 , \pi \geq 0 \right\}$$

Let, $E_{b_2}\left(F(x,b_2)\right) = F(x) = Q(x) = E_{b_2}\left(Q(x,b_2) \right)$.

The equivalent deterministic program for the problem (3.2), then, becomes

$$\min_{x \in K} \quad (cx + Q(x))$$

which is obtained in [35,78,77,146,148] .

(ii) Using any of the decision criteria described above such as replacing random variables by their expectations [76,99,123] or applying the aspiration or α-fractile criterion [57,58] , one can get the equivalent deterministic program.

(iii) By reducing the stochastic programming problem to another equivalent deterministic form [44,45,46] or introducing another problem called the surrogate problem [92] .

(iv) Using the generalized inverse or pseudo-inverse of matrices to reduce the original problem into a constrained generalized median or hypermedian problem [23,28] or into another deterministic program [36,73] .

The equivalent constrained generalized median problem as obtained in [23] can be expressed as follows:

$$(3.4) \qquad \text{Min} \quad (c - c_2 A_{22}^{+} A_{21})x + E\left[\max_{s=1,2,\cdots k} w_s(A_{22}^{+}(A_{21}x - b_2))\right]$$

$$\text{subject to} \quad A_{11}x = b, \quad x \geq 0$$

where A_{22}^{+} is a generalized inverse of a matrix A_{22} and w_s are the k-extreme points of the convex polyhedron defined by the constraints

$$w\,P = c_2 P, \; w \geq 0$$

where $P = I - A_{22}^{+}A_{22}$ is the projection on to the null space of A_{22}.

The objective functions of these deterministic programs are convex and continuous in the interior of K, [28,34,35,36,139,146,148,149] and seperable also if the distributions are discrete [44,45,46] .

Optimality Conditions:

The equivalent deterministic programs for the stochastic programming problems considered in various papers have convex objective functions which are not necessarily differentiable. The necessary and sufficient conditions for the optimality and duality of such programs are obtained in [35,43,143, 145,146] by constructing the supporting hyperplanes for the objective functions.

If the objective function of the deterministic program is convex and differentiable as obtained in [145], then the Kuhn-Tucker conditions (necessary and sufficient) are applied to get the optimality results.

Different types of stochastic programming problems such as the descrepancy cost program with (i) linear loss (ii) quadratic loss (iii) linear compensation (iv) constrained linear loss and (v) piecewise linear loss are discussed in [36] where some necessary and sufficient conditions for the optimality of these programs are obtained.

The optimality and duality relations are also obtained seperately for the complete problem in stochastic programming expressed in the following form [148,152]

$$\begin{aligned} &\min \quad cx + E_{b_2}\left[q^{+}y^{+} + q^{-}y^{-} \right] \\ (3.5) \qquad &\text{subject to} \quad A_{11}x = b \\ &\qquad\qquad A_{21}x + Iy^{+} - Iy^{-} = b_2 \\ &\qquad\qquad x, y^{+}, y^{-} \geq 0 \end{aligned}$$

which has the continuous distribution function for the random variables involved. Also, for the discrete distribution case, these conditions are obtained in [44,45,46] .

As it is shown in [147] that the solution set K of the problem (3.2) is convex polyhedron and in [85] that the optimal solution x of the problem (3.2) has components at positive level corresponding to the mutually independent columns of the basis matrix and the similar results for the n-stage program obtained in [28,85] , it follows that the optimal solution of the problem will lie on one of the vertices of K.

In [110] , a nonlinear programming problem with randomized solution is considered and the saddle point theorem and related optimality conditions are obtained similar to the Kuhn-Tucker conditions [62] , with a slight restrictions on the shape of the optimising constraints.

Problems considered in [52,77,139] are defined and discussed in the abstract vector spaces. The problem considered in [52] is defined in an $L-\infty$ space. The characterization of the dual problem and sufficient conditions for the equivalence of the existence of the optimum and that of the non-negative saddle point for the associated Lagrangian function are established. The saddle point theorem and the duality for the problem defined over the Hilbert spaces, are obtained in [77] . The weak and strong duality results are established for the problem defined over the Banach spaces in [139] .

Inequalities in Stochastic Programming:

Some bounds for the values of the objective functions of the stochastic programming problems are obtained in [5,6,75,79,80,130,131] . Madansky [75] and Vajda [130,131] have obtained these bounds for the linear case while Mangasarian and Rosen [80] , Mangasarian [79] and Bui Trong Lieu [5,6] have considered for the nonlinear case. The important results are the following in the form of inequalities:

(3.6) $E\ \nu(b,\bar{x}_{Eb}) \geqslant \min_x\ E\ \nu(b,x) \geqslant E\ \min_x\ \nu(b,x) \geqslant \min_x\ \nu(Eb,x)$

where $\nu(b,x) = \min_y \{ \phi(x) + \psi(y) / g(x) + h(y) \geqslant b \}$, ϕ, ψ and each component of the vector valued functions $-g$ and $-h$ are convex and continuous functions of their respective arguments and $\bar{x}_{Eb}$ is the solution of the problem $\min_x \nu(Eb,x)$. If the problem is that of maximising a profit function, the bounds are also obtained in [79].

The consideration of the inequalities (3.6) in the abstract vector spaces is given in [5,6].

Single-Stage Programming Under Risk.

The random variables appearing in the problems considered in this section are known completely by their distributions. In most of the problems, the random variables are assumed to be normally distributed [9,51,59,136,138]. In some of these problems a risk preference functional is optimised as a deterministic programming model, the optimal solution of which gives the optimal solution for the original problem [136,138].

The problems of different characteristics such as the minimum risk-solution problems, the problems considered as games against the nature, chance-constrained programming problems etc. are studied in this field of stochastic programming. Since, a good deal of literature on chance-constrained programming has been developed, such problems will be studied in the next section.

The minimum risk solutions are considered in [9,59,137].

A solution x_u of the problem $\max_{x \in X} \phi(x,u) = P[w / c'(w)\, x \leq u]$

where u is a given number, P denotes the probability, $c(w) = c^0 + \sum_{i=1}^{r} c^i t_i(w)$, $t(w) = (t_1(w), \ldots t_r(w))$ is a bounded random vector on the probability space $(\Omega, \mathcal{F}, P_r)$, and X is the non-void bounded set $\{x / Ax = b, x \geqslant 0\}$, is called the u-min risk solution. For $r = 1$, and t_1 having a continuous distribution $F_{t_1}(.)$ with $c_1 > 0$, the solution x_u does not depend upon the distribution $F_{t_1}(.)$ [9]. It is further shown that if $t_i(w)$ are independently normally distributed with means m_i and variances σ_i^2, $i = 1,2,\ldots r$, and if $0 \notin X$, then x_u is the solution of the problem $\max_{x \in X} \phi(x,u) = (u - \sum_{i=1}^{r} m_i x_i) / \sum_{i=1}^{r} \sigma_i^2 x_i^2$. If $m_i=0$, x_u is independent of u and can be obtained by solving the quadratic programming problem.

In [59] an approximate formula for the expectation and the variance of the min risk solution in the stochastic programming problem with random variables being normally distributed is obtained under the assumption that the linear programming problem has the optimal base.

With a given set of parameters if the optimal bases for the primal and dual problems are given where parameters are subject to stochastically variations, then these bases remain optimal throughout the certain subregion of the parameter space and are called the optimal regions for the respective bases [137].

For the problems considered in [136,138], the truncated minimax criterion is applied to obtain the deterministic equivalent programs. In [138], attaching weights 1 and m to expectations and standard deviations respectively, some properties including the duality theorem

have been discussed. In [136], it is shown that the optimal dynamic path comes out to be a geodesic in the Reimannian space with the standard deviation of the objective function as the metric. The important result is that the optimal path obtained is independent of the risk attitude adopted.

In [67, 135], the stochastic programming problems are treated as games against nature. The set X of feasible strategies are the enterpreneur's strategies and the points in the parameter space P are the nature's strategies [135]. The two cases are considered in [67], (i) the joint distribution of the random variables is known (ii) the joint distribution of the part of random variables is known. Then it is shown that the problems reduces to the two person-zero-sum game (and a statistical game) for both the case (i) and (ii).

<u>Computational Procedures</u>.

Several computational procedures are developed and several are advised for solving equivalent deterministic programs obtained in [28,14,35,44,45,50, 56,57,58,76,94,138,140,144,148]. A brief survey of these solution methods is presented here.

Three methods of solutions for the problem min cx, subject to $Ax \geqslant b$, $x \geqslant 0$, are discussed in [76]. These are (1) The 'expected value solution'. This method replaces the random variables in the above problem by their expectations. If the solution to this problem is 'permanently feasible' i.e. $P[Ax \geqslant b] = 1$ is satisfied, then x is used as the solution for the original problem,(2) 'The fat solution'. If the distribution is discrete, then a procedure indicates to obtain a 'pessimistic'

values of the random variables such that the condition of permanent feasibility is satisfied.(3) 'The Slack Solution'. The method replaces the single-stage problem by the two-stage complete problem (3.5).

Computational procedures developed in [35] depend upon the optimality conditions of the problem and the supporting hyperplane constructed for the purpose.

A cutting hyper-plane method which is shown equivalent to the partial decomposition algorithm for the dual problem is applied to solve the two-stage stochastic programming problems [140].

An algorithm for the complete problem (3.5) is obtained in [148] for the case where random variables are continuously distributed.

A methodology is proposed in [94] for finding the solution of the resource allocation problem under uncertainty. The method applies the Lagrange's technique of undetermined multipliers.

By making the informal estimates of the certainty equivalents for the demand, a method of solving the resulting linear programming by ordinary activity analysis is considered in [144] for obtaining the approximate solution to the original stochastic programming problem. An algorithm for the linear homogeneous programming problem is developed in [138]. This is an iterative sequence of linear programs pursued upto any accuracy level. Analgorithm for the parametric problem which is equivalent to a vector maximum problem which, in turn, is equivalent to a stochastic programming problem is obtained in [57,58].

Some methods of solving the transportation problems are given in [50,56,109,142]. It is indicated in [44,45,46] that the network theory can be of great help to solve the stochastic programming problem which

can be reduced to a deterministic program representable in the network form.

Applications

Applications of stochastic programming to various realistic problems is indicated here. The stochastic programming theory has been applied to the transportation problems [44,45,50,56,100,109,142] and the assignment problems [73,132,151] . The various algorithms for solving these are presented in the above references. A relationship between the transportation problem and the assignment problems is also considered.

Analytic decision rules and horizon rules are developed for the stochastic Warehousing problem in [24] . The problem is solved first by backward induction and then by the forward working algorithm. Stochastic programming results are also applied to the Leontief production model [44,45] . Production inventory and the dynamic inventory problems are getting increased application of the stochastic programming models of two-stage formulations [44,45,46,35,80] .

The capital investment and the investment in the stock-market are discussed in [80,90] . The problem of allocating the fixed amount of resources among the competing sectors of enterprise, the operations of which may be inter-related, is considered in [94] . The application of stochastic programming problems to the agricultural problems and the portfolio selection problems are shown in [138] . The results of stochastic programming problems are also applied to the stochastic control problems [139] . Several other applications are cited in [62] .

(3-ii) Chance - Constrained Programming

The concept of chance-constraints [15] was first introduced in 1958 as one of the constraining models for scheduling the production of heating oil under random demands with known distributions. A model of chance-constrained programming was, then, developed by Charnes and Cooper [16] and was further considered by them and others [17 to 22,25,26, 27,30,31,32,47,63,68,69,71,72,84,96,97,98,106,112,113] to deal with the uncertainties present in the model.

A general chance-constrained model can be expressed as follows:

$$\text{Maximise} \quad f(c,x)$$

$$\text{subject to} \quad P\left[\sum_{j=1}^{n} a_{ij}x_j \leq b_i\right] \geq \alpha_i, \; i=1,2\text{——}m \tag{3.7}$$

where P denotes the probability, α_i's are the pre-assigned high-probabilities and some or all elements in c, $[a_{ij}]$, b are random variables. The constraints $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$, i=1,2,——m in (3.7) may be violated some times but not more than $100(1 - \alpha_i)\%$ of the time.

Decision rules in Chance-Constrained Programming

Decision variables in (3.7) are functions of the random parameters involved in the problem i.e. they can be expressed as $X = g([a_{ij}], b, c)$. These decision-variables are generally known as decision rules in the chance-constrained programming. According to their dependence on the random parameters, they can be characterized as the zero-order, linear order and multi-period decision rules.

Any decision variable X which is not a function of the random parameters appearing in the problem explicitly is called the zero-order

(or constant) decision rule [22,25,27,68,84] . If the matrices $[a_{ij}]$ and c have constant elements, then the decision vector X expressed linearly in terms of random b, is called the linear order (or linear) decision rule [15,16,17,19,22] . In the multi-stage models (i.e. general n-period models) X_j represents the vector of decision rules for the j-th period and X_j is considered as the function of the previous decisions taken in periods 1 to j - 1 but not of those (though implicitly reflecting these) which are still to be taken in periods j to n. Such a decision vector is known as the multi-stage or n-period decision rule [26,31,71,97] .

In the multi-period models (or even in the one period models) of chance-constrained programming, there may exist values of the random variables in periods 1 to j such that for these values X_j(or X) becomes inconsistent. To overcome such difficulties several procedures are developed in [26] . For each of these procedures the optimal rule X_j can be reduced to finding the optimal rule over the sample points for which the constraints are consistent.

Chance-Constrained Models:

Chance-constrained programming problems have mainly three types of objective functions. These are (i) E-type, (ii) P-type and (iii) V-type [19.20] . Also, the chance-constraints of the problem may be satisfied either (i) under conditional probability or (ii) under total probability. The chance-constraints may also be satisfied jointly, in which case they are called joint chance-constraints.

The E-type model is that in which the expected value of the objective function is optimised. A general n-period E-model can be formulated as follows

$$\text{Max} \quad E\left(\sum_{j=1}^{n} C_j^T X_j\right)$$

$$\text{(3.8)} \qquad \text{subject to} \quad P\left(\sum_{j=1}^{i} A_{ij}X_j \leq b_i\right) \geq \alpha_i, \quad i=1,2,\cdots,n \quad j=1,2,\cdots,n$$

$$X_j \geq 0$$

where E and P denote the expectation and probability operators respectively. All the matrices A_{ij}'s are assumed constant. X_j is the jth period decision rule which is a function of b_i, c_i, i=1,2,---j-1 but not of b_i, c_i, i=j, j+1,---n. It is assumed that the random variables b_i, c_i, i=1,2---n have known joint distribution function. The constraints (3.8) are known as the 'conditional chance-constraints' and can be expressed as

$$\text{(3.9)} \qquad \sum_{j=1}^{i} A_{ij}X_j \leq F_i^{-1}(1-\alpha_i)$$

where $F_i^{-1}(1-\alpha_i) = \max\{y \mid F_i(y) \leq 1-\alpha_i\}$ and $F_i(.)$ is the conditional distribution of b_i for given b_k, c_k, k=1,2,----i-1.

The other way of the interpretation of (3.8) is

$$\text{(3.10)} \qquad \int F_i\left(\sum_{j=1}^{i} A_{ij}X_j\right) dG_{i-1}(b_1,\text{---}b_{i-1},\ c_1\text{---}c_{i-1}) \leq 1-\alpha_i$$

where G_{i-1} is the joint probability density function of all the random variables b_k, c_k, k=1,2,----i-1. This is, then, called the total probability chance-constraint.

In [26,72] a necessary condition for the optimal decision rule for the choice of X_j in the E-model with the conditional chance-constraints is obtained by different methods such that X_j comes out to be the piecewise linear in the conditional fractile points $F_j^{-1}(1-\alpha_j)$, X_{j-1},----X_1.

A very similar result for the total chance-constraint model is established in [26] .

The P-model is that in which we maximize the probability that atleast a certain level of the value of the objective function is achieved subject to chance-constraints and the choice of the decision rules from a given class. This type of model can be expressed as follows:

$$\text{Max} \quad P\left[\sum_j C_j^T X_j \geq C_o^T X_o = K\right]$$

$$(3.11) \qquad \text{subject to} \quad P\left[\sum_{j=1}^{n} A_{ij} X_j \leq b_i\right] \geq \alpha_i, \; i=1,2,\text{---}n,$$

$$P\left[X_j \geq 0\right] \geq \beta$$

This model has been considered in [19,20,31,106] . In [31] , it is shown that the results obtained in [26,72] provide sufficient conditions for the optimality of decision rules in the n-period P-model (3.11) for conditional chance-constraints as well as for the total probability chance-constraints. It is also shown there that an n-period P-model can be converted into (n+1)-period E-model of the special type for which the results of [26,72] can be applied. In [58] the aspiration criterion is used to find the deterministic program for the P-model.

Much of the work in chance-constrained programming [15,16,17,18,19, 68,69,106] is centered on finding out the deterministic equivalent 'linear and nonlinear' programming problems for the various chance-constrained models. In all of these problems the decision rule is restricted to the linear decision rule i.e. X = Db, where D is a constant matrix, and the random variables are restricted to the normal distributions.

The V-model of chance-constraint programming can be expressed as follows:

$$(3.12) \qquad \operatorname{Min} \; E(c'x - c^{*}x^{*})^2$$

subject to some chance-constraints. The deterministic form of this model has been obtained in [17,18,106] for the case where random variables are normally distributed.

The joint chance-constraint expressed in the following form

$$(3.13) \qquad \prod_{i=1}^{n} P\left[\sum_{j=1}^{n} a_{ij}x_j \leq b_i)\right] \geqslant \beta$$

has been considered in [29,84]. A case of random variables being multi-normally distributed is also considered in [84]. A deterministic model was obtained their.

<u>Rejection Regions in Chance-Constrained Programming</u>

The concept of rejection regions is introduced in [29] where it states that the rejection region for the ith constraint in the constraining system (3.7) will be the set of sample points $(b_1, b_2 \text{——} b_r)$ for which $\sum_{j=1}^{n} a_{ij}x_j^{*} > b_i$, where x_j^{*}, $j = 1,2,$——n are specified optimal decision rules. This can be interpreted as the set of sample points for which the subsystem is violated by a specific optimal vector of decision rules, i.e., if the rejection region is denoted by R_i for the ith constraint then all the constraints of the type

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i, \quad b_i \in R_i$$

can be rejected from the problem and only those constraints for which $b_i \in R_i^{c}$ (the complement of R_i) are considered in the problem. In [29],

it is shown that the rejection region for the problem under consideration is the set of sample points for which $Z^* \in H_{Z^*}^{-1}(1-\alpha)$ where $Z^* = C X$ for the optimal X, and H_{Z^*} is the distribution function of Z^*. Similarly, the rejection regions for the E-model and P-model of the chance-constrained programming are obtained. In [98], the application of the rejection region theory is made in the analysis of the solutions in the problem of the two-stage chance-constrained programming.

Chance-Constrained Games

The field of chance-constrained games is new in the field of chance-constrained programming. The following programs for the zero-sum two person game are considered for the two players in [96].

$$\begin{aligned} &\text{Max } \delta \\ &\text{such that} \\ (3.15) \qquad & P\left[\min_q \ p^T A q \geq \delta\right] \geq \alpha \\ & p^T e_m = 1, \ p \geq 0 \\ & q^T e_n = 1, \ q \geq 0 \end{aligned}$$

$$\begin{aligned} &\min \ \rho \\ &\text{such that} \\ & P\left[\max_p \ p^T Q q \leq \rho\right] \geq \beta \\ & p^T e_m = 1, \ p \geq 0 \\ & q^T e_n = 1, \ q \geq 0 \end{aligned}$$

where e_m and e_n are sum vectors, α and β are the prescribed probabilities and decision rules p for (3.14) and q for (3.15) are zero-order decision rules. Matrix A has random elements.

The programs (3.14) and (3.15) represent the zero-sum game with zero-order decision rules so that they are called the zero-zero chance-constrained game models. The deterministic models for (3.14) and (3.15) are obtained. The saddle point is derived for the stochastic game where the two players play with optimal actions to their different deterministic games. The two important properties are obtained, (a) the optimal actions for both players correspond to their different deterministic zero-sum games, though they are playing the same chance-constrained zero-sum game and (b) $\delta \leq \min \rho$ for $\alpha, \beta > \frac{1}{2}$ and for any feasible solution to (3.14).

The same models (3.14) and (3.15) are considered in [32] with matrix A of constant elements and with the assumption that players are not able to specify their own strategies completely. Such types of games are called chance-constrained games with partially controllable strategies. In deterministic form, these become the two person non-zero sum games.

Solution Methods for the Deterministic Models.

Several authors [69,84,97,112,113,111] have indicated the solution methods and solved some examples for deterministic equivalent models of chance-constrained programming. Kataoka [69] suggested a method with the help of Kuhn-Tucker conditions. In [84], the generalized linear programming techniques are applied for the equivalent deterministic program after taking the logarithmic transformation of the joint constraint (3.13).

In [112,113], the equivalent deterministic form: max $z(x)$: subject to $y^k(x) \leq 0$, $F(y^k(x)) = \alpha_k$, where the $F(x)$ is the distribution of $y^k(x)$, is obtained for the chance-constrained model. max $z(x)$ subject to $P[y^k(x) \leq 0] \geqslant \alpha_k$, where $y(x) = Ax-b$ is a random vector. The method for solving the joint chance-constrained model with only b random, is developed

by transforming the linear programming problem of the form: Max cx, subject to $Ax = b$, $x \geq 0$ into the cost-unitized linear programming problem of the form: Max $1y$, subject to $A T^{-1} y = b$, $y \geq 0$, where 1 is the sum vector.

In [97], it is indicated that some computational technique with the help of decomposition method can be obtained to solve the equivalent deterministic program for the two-stage (and n-stage) programming under uncertainty with chance-constraints.

Applications.

A survey of the applications of chance-constrained programming problems is made in [71]. Here, we indicate only some of them. The financial planning problems, such as capital budgeting, investment problems, stock-marketing investing and planning management investments in research and development and saving loan models etc. are considered in [22,25,62,90, 98].

Chance-constrained models are utilized in the network and the generalized network problems considered in [21,27]. The industrial problems such as the scheduling the production of heating oil, the problem arising in the metal melting furnace, pricing policy for the developing management for analysing the pricing of rail road in the presence of competition, cattle feed problems and others are getting the use of chance-constrained models as given in [15,30,47,133] and other reference cited in [71].

(3-iii) Distributional Problems

Stochastic programming problems in which the main emphasis is given on finding the distributions, means and variances of the optimum values of the objective functions and/or solutions where random variables appearing in

the problems are known by their distributions, are called the distributional problems. This name was suggested by Vajda in [131]. Madansky [75] introduced such problems as 'wait and see' problems by which we mean that the decision vector is chosen after the random variables have been observed. Tintner [124] was the first to introduce such a problem of stochastic programming in 1955. This programming model was then developed by Tintner and others [2,7,11,12,13,101,102,125].

Tintner [124] considered the problem of minimizing $b'x$ subject to $Au = \hat{c}$, $u \geq 0$, where $\hat{c}$ is an observation on the random variable c. In this problem it was required to find the cumulative distribution function of the optimum of this problem where random variables have known distributions.

Two approaches called 'passive' and 'active' introduced by Tintner have generally been considered by various authors. In the passive approach, the probability distribution of the objective function is derived explicitly or by numerical approximations where decision rules depend upon this distribution. For all admissible situations, the conditions of the simple non-stochastic linear programming are satisfied and, thus, the optimal is achieved. The 'active approach' to the stochastic linear programming can be specified as follows:

$$\begin{aligned} &\text{Maximize} \quad z = c'x \\ (3.16) \qquad &\text{subject to} \quad AX \leq BU, \quad x \geq 0 \\ &u_{ij} \geq 0, \quad \sum_{j=1}^{n} u_{ij} = 1 \end{aligned}$$

where X is a diagonal matrix with elements x in the diagonal and B is a diagonal matrix with elements b in the diagonal. The problem of distribution

of $\{\max z\}$ will depend upon the allocation matrix $U = [u_{ij}]$ which defines a set of control (or decision) variables which may be appropriately chosen to optimize risk preference functional (i.e. the utility function) associated with the objective function.

Some relations between the active approach and the passive approach are discussed in [101,102] . Also, some inequalities for the optimal values of the problem for these two approaches are derived for the case where the coefficient-matrix of the constraints is random. Based upon the properties of the distributions of the optimal values and other order statistics, some approximate bounds are also obtained for the passive approach which can be easily extended to the active approach [101] . Under some operational methods the numerical approximations of the probability distribution of the objective function are derived in [102] .

Babbar [2] presented an approach for approximating the certain properties of the distributions of the values of x in the solutions of the model

(3.17) Optimize $(C + c)'x$

subject to $(B + b)x = Q + \epsilon$, $x \geqslant 0$,

where B, Q, and C are constant mxn, mx1, and nx1 matrices, b is mx1 random vector, ϵ and c are the corresponding error vectors. Wagner [141] modified the method given in [2] and developed an outline of the application of linear programming under uncertain conditions.

In [7,11,12,13] the problems considered are similar in nature but different in character. The determination of the distribution function of the minimum of the objective function for a linear stochastic programming problem depending on a single random variable with known distribution is

discussed and obtained in [7] . In [11] , the cum-ulative distribution functions, means andvariances are obtained for the following random variables

(3.18) $$\delta_1(t) = \min_x \left\{ c(t)\, x / x \in X,\ t \in T \right\}$$

(3.19) $$\delta_2(t) = \min_x \left\{ cx / x \in X(t),\ t \in T \right\}$$

and

(3.20) $$\delta_3(t) = \min_x \left\{ c(t) x / x \in X(t),\ t \in T \right\}$$

where $X = \left\{ x/Ax = b,\ x \geqslant 0 \right\}$, $X(t) = \left\{ x/Ax = b(t) = b^o + \sum_{i=1}^{r} b^i t_i, x \geqslant 0 \right\}$

and $c(t) = c^o + \sum_{i=1}^{r} c^i t_i$ and $t = (t_1, t_2 \text{——} t_r) \in T$ is only random variable with known joint probability distribution function defined in the bounded region T. These results are also obtained for the problem where A is the matrix of random elements [13] . Using the passive approach the Laplace transform is applied in [12] in order to derive the above results and certain conditions for the problem (3.20).

Decision Analysis:

Decision analysis has been developed in [10,14] for the linear stochastic programming problems having the random variables in the objective functions only. Decision regions for the problem (3.18) where $T = \left\{ t/\ \delta_j \leq t \leq \delta_j',\ 1 \leq j \leq r \right\}$ is only a random vector, are derived and discussed in [10] . If X as defined in (3.18) is bounded having all bases non-degenerate, then a convex polyhedral set $S_i \subset T$ $(1 \leq i \leq p)$ called the 'regions of decisions' for the problem (3.18) associated with the set of basic solutions x_i and basis B_i and a set of (r+1) vector

$\alpha_i = (\alpha_{oi}, \alpha_{1i}, \text{——} \alpha_{ri})$ are obtained such that

(i) $\bigcup_i S_i = T$, (ii) $t \in S_i \Rightarrow \delta_i(t) = c(t)\,\hat{x}_i = \alpha_{oi} +$

$\alpha_{1i} t_1 + \cdots\cdots + \alpha_{ri} t_r$ and (iii) $i \neq j \Rightarrow P\left[t/t \in S_i \cap S_j\right] = 0$

where P denotes the probability. The distribution function and the mean of $\delta_1(t)$ and the probability that the given basis is optimal are obtained in terms of probability density functions of t, α_i and S_i. Similar results are also obtained by duality with b taken as random variables b(t) defined as above.

The statistical decision analysis for the stochastic linear programming problem with random variables only in the objective function is studied and discussed in [14] . The expected value of the perfect information are derived with the help of the distribution of the optimum value of the said problem. A case where the random vector has the multi-normal distribution is also considered and a discussion is made of the dual problems related to the stochastic linear programming problems. Procedures are also developed for finding out the numerical results.

Convergence and Stability Analysis of Solutions

The convergence of the distribution problems and their solutions, recursive linear programming problems and their stability and the stability of the solutions and the optimal values in the stochastic programming problems are considered in [1,91,103,104,105,128,129] .

A linear programming problem with all its parameters random is considered in [91] . Consider the sequence of random vectors and matrices where the maximum variance of the random elements converges to zero. Then, under the assumption that optimal base is non-degenerate and under certain other conditions the random programming problem comes out in limit to be normally distributed.

Considering the feasibility, optimality and duality for the deterministic program of the stochastic programming problem over the sets of distinct and selected points in the parametric space, a generalization to the case of stochastic program is shown by utilizing the convergence of these distinct and selected points in the par metric space having the non-degenerate regions to the limit points [128] .

Recursive linear programming defined as the sequence of linear programming problems in which the recursive relation made through the parameters involved in the problem, is studied in [129] where only the right-hand-side parameter of the linear system of inequalities is related recursively. The stability of this recursive programming is discussed and analyzed and results obtained are extended to the active approach of the stochastic linear programming.

The sensitivity and stability analysis for the solution of the problem (3.17) with inequalities in-place of equalities is discussed in [105,104] where in [105] the analysis of the stability for the truncated solutions for the passive and active approaches in the stochastic linear programming problems is considered. A problem of minimizing $f(x)$ subject to D: $Ax = b$, $x \geqslant 0$, with b a random vector having known means and variances, is said to be stochastically stable modullo $\in$, if, for the fixed vertices of the random polyhedron D (defined above) determined by the solution of the problem obtained after replacing b_i by its mean, the minimum of the problem is attained with probability greater than $1 - \in$ [1] . For the stochastic stability optimal is obtained with probability greater than $1 - \in$. Using the Chebyshev inequality a sufficient condition for the stability of the optimal solution is established.

Applications of Distributional Problems.

In [93], the linear stochastic programming is applied in obtaining the team decisions where pay-off to the team is a convex polyhedral function of the decision variables i.e. pay-off function is random variable.

In development planning problems the stochastic programming problems are getting increased applications, see for example [126, 127]. A stochastic programming model has been applied to the India's third five year planning model in [127]. Also, in the static and dynamic model of the two sector development planning the stochastic problems are applied [102].

Transportation and assignment problems are also getting use of the stochastic Programming problems [8,12,132]. In [7], the input output analysis of a single farm producing three crops with three reasources is discussed.

4. SUMMARY OF THE THESIS

Stochastic linear programming problems have been studied extensively in the literature. A few cases of nonlinear stochastic programming problems have also, been considered. In the present work, mostly nonlinear stochastic programming problems are studied. Here, the development of the work is theory-biased.

In stochastic programming problems, some important inequalities relating the bounds of the problems, have been obtained by various authors [5,6,75,79,80,130,131] for linear as well as nonlinear cases. Our work in Chapter II presents some more development in this direction. The problem considered is that of finding the maximum expected profit or that of expected maximum profit subject to some random constraints. The problem

studied is the following:

Find the sup. of $\phi(c(w),x)$

subject to $g(a(w),x) \leq b(w)$

where $x \in X$, an abstract vector space, $w \in \Omega$, the joint sample space, $p: (\Omega, \mathcal{F}, P) \longrightarrow W$, a vector space, where the probability space $(\Omega, \mathcal{F}, P)$ is known.

For the parametric version of the problem, it is shown that if ϕ is a convex (or quasi-convex) function of $c(w)$ (or $c(w)$ and x or x only) the function $\alpha(p(w)) = \sup_{x \in B_{p(w)}} \phi(c(w),x)$ is a convex (or quasi-convex) function of $c(w)$ (or $c(w)$ and $b(w)$ or $b(w)$ only) where $p(w) = (c(w), a(w), b(w))$ and $B_{p(w)} = \{x / g(a(w),x) \leq b(w),\ x \in X\}$. These results help in obtaining some of the important bounds for the stochastic programming problems.

For the stochastic version of the problem, the following are the important inequalities:

$$\text{(i)} \quad E \sup_{x \in B_{p(w)}} \phi(c(w),x) \geqslant E_{ab} \sup_{x \in B_{E_c p(w)}} E\,\phi(c(w),x) \geqslant E\,\phi(c(w),x_{E_c})$$

$$\geqslant E_{ab} \sup_{x \in B_{E_c p(w)}} \phi(Ec(w),x)$$

and

$$\text{(ii)} \quad E \sup_{x \in B_{p(w)}} \phi(c(w),x) \geqslant \sup_{x \in B_{Ep(w)}} E\,\phi(c(w),x) \geqslant E\,\phi(c(w),x_{Ep(w)})$$

$$\geqslant \sup_{x \in B_{Ep(w)}} \phi(E(c(w),x)$$

where the convexity of ϕ is assumed for the last inequalities of (i) and (ii), $B_{Ep(w)} = \{x/g(Ea(w),x) \leq Eb(w),\ x \in X\}$, similarly $B_{E_c p(w)}$ with E replaced by E_c, E, E_c and E_{ab} denote the expectations taken with respect to p(w), c(w) and a(w) b(w) respectively, where $x_{Ep(w)}$ is the solution of the problem $\sup_{x \in B_{Ep(w)}} \phi(Ec(w),x)$.

One can easily see that the inequalities (i) are the natural generalizations of those obtained in [79] .

Some bounds for the fat solutions of the problem are also obtained. It is remarked there that the above inequalities also hold for the case where b's are the convex functions of c's or c's are the convex functions of b's.

Further, if ϕ is a concave function of c(w), the result obtained in another theorem states that the first two inequalities of (i) are satisfied as they are, the third becomes $E \sup_{x \in B_{E_c p(w)}} \phi(Ec(w),x) \geqslant E \sup_{x \in B_{p(w)}} \phi(c(w),x)$ where $Ec(w) > 0$. Several examples are furnished to justify the above results.

In Chapter III, the duality and saddle point theorems for the quadratic programming problems under uncertainty are established. Madansky [77] established the above results for the linear case with the help of Theorem V.3.2 of [64] . The formulation of the problem here is similar to that of Madansky [77] differing in one or two places only. The following problem is studied:

$$\underset{x \geqslant 0}{\text{Minimise}} \left[p'x + \tfrac{1}{2} x'Cx + E \underset{y \geqslant 0}{\min} \left\{ q'y + \tfrac{1}{2} y'Dy \right\} \right]$$

subject to $Ax + By \geqslant b$, almost all b

$$x \in E^r, \; y \in E^s$$

where b is the random vector with known distributions, C and D are non-negative definite symmetric matrices. After reformulating the problem one obtains

$$(4.1) \quad \text{Min} \int_{\mathcal{B}} c' \xi(b)\, d\mu(b) + \tfrac{1}{2} \int_{\mathcal{B}} \xi'(b)\, Q\, \xi(b)\, d\mu(b)$$

subject to $H\, \xi(b) \geqslant b$, $\xi(b) \geqslant 0$ all b

where $H = [A, B]$, $c' = (p', q')$, $\xi'(b) = (x', y'(b))$, $Q = \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}$ and $\mu(b)$ is the probability measure defined on the probability space $(\mathcal{B}, \mathcal{F}_1, \mu)$.

The dual of (4.1) can be expressed as follows.

$$(4.2) \quad \text{Max} \int_{\mathcal{B}} b' \delta(b)\, d\mu(b) - \tfrac{1}{2} \int_{\mathcal{B}} \xi'(b)\, Q\, \xi(b)\, d\mu(b)$$

subject to $H' \delta(b) - Q\, \xi(b) \leqslant c$, $\delta(b) \geqslant 0$ almost all b.

Let $[\xi(b)]$ be the collection of all those $\xi(b)$ whose first r-coordinates are the coordinates of x. Let $[\delta(b)]$ be the collection of all δ's indexed on b. Assume that $[\xi(b)]$, $[\delta(b)]$ and b are measurable and square integrable with respect to μ. After defining the inner product of the two collections $[\xi(b)]$ and $[\eta(b)]$ of the same space such that $\langle [\xi(b)], [\eta(b)] \rangle = \int_{\mathcal{B}} \xi(b)\, \eta(b)\, d\mu(b)$, and similarly for $[\delta(b)]$ and $[\theta(b)]$ in the same space, the problems (4.1) and (4.2) are then defined on the Helbert spaces. The Lagrangian function for the problem (4.1) can be expressed as follows

$$\Phi(\xi(b), \delta(b)) = - \int_{\mathcal{B}} c' \xi(b) d\mu(b) - \tfrac{1}{2} \int_{\mathcal{B}} \xi'(b)\, Q\, \xi(b) d\mu(b) + \int_{\mathcal{B}} \xi'(b) H' \delta(b) d\mu(b) - \int_{\mathcal{B}} b' \delta(b) d\mu(b).$$

The saddle point theorem and the duality theorem are then established by two different techniques: (i) with the help of fundamental (Hahn-Banach) seperation theorem and (ii) by the method corresponding closely to that of. Albert Madansky [77]. The optimality criteria for this problem and that of the second stage program are also provided.

Chapter IV contains the discussion of a class of stochastic programming problems. The importance attached to these problems is that all these problems generate the similar type of decision equivalent deterministic programs. The stochastic problems studied are:

(a) Stochastic linear programming problems with linear losses

$$\underset{x \geq 0}{\text{Minimize}}\ E\left[c'x + \underset{y \geq 0,\ z \geq 0}{\text{Minimize}} \left\{d'y + f'z + \|Dz\|\right\}\right]$$

subject to

(4.3) $Ax \geq b$,

(4.4) $P\left[Tx + Wy \geq a\right] \geq \alpha$

(4.5) $P\left[Mx + Nz \geq q\right] = 1$

(b) Stochastic linear programming problems with quadratic losses

$$\underset{x \geq 0}{\text{Minimize}}\ E\left[c'x + \underset{y \geq 0,\ z \geq 0}{\text{Min}} \left\{d'y + f'z + \|Dz\|^2\right\}\right]$$

subject to (4.3), (4.4) and (4.5)

(c) Quadratic Programming problems under risk with linear losses

$$\underset{x \geq 0}{\text{Minimize}}\ E\left[c'x + \tfrac{1}{2}x'Sx + \underset{y \geq 0}{\min} \left\{d'y\right\}\right]$$

subject to (4.3) and (4.4)

and

(d) Stochastic linear programming problems with linear losses subject to linear constraints and a quadratic constraint

$$\underset{x \geq 0}{\text{Minimize}}\ E\left[c'x + \underset{y \geq 0}{\min}\{d'y\} + \underset{z \geq 0}{\text{Min}}\{kz\}\right]$$

subject to (4.3), (4.4) and $\ell z + rx - \frac{1}{2}(x'\Gamma x) \geqslant \lambda$

where $c, x, r \in R^n$, $d, y \in R^{m_1}$, $\ell, f, z, k \in R^{m_2}$, $b \in R^m$, $a, \alpha \in R^{m_1}$, $0 \leq \alpha_i \leq 1$, $i = 1, 2 \cdots m_1$; q, $1 \in R^{m_2}$, $1 = (1, 1, \cdots 1)$ m_2-tuples, P and E denote the probability and the expectation respectively. The matrices A, D, T, W, M, N, S and Γ are appropriately defined where S and Γ are positive semi-definite symmetric matrices. In these problems, c, q, a, M and T are random vectors and matrices respectively. All the random variables are known by their distributions. The random variables in T and a are normally distributed or they have known means and variances.

Under the suitable formulation the problem (a) has its decision equivalent deterministic program

$$(4.6) \qquad \text{Min}\ h'x + \sum_{i=1}^{m_1} (x'Q^i x)^{\frac{1}{2}} + E\,\|G\,(q - Mx)\|$$

subject to $Ax \geqslant b$, $x \geqslant 0$

and problems (b), (c) and (d) have the equivalent form:

$$(4.7) \qquad \text{Min}\ h'x + \sum_{i=1}^{m_1} (x'Q^i x)^{\frac{1}{2}} + \tfrac{1}{2}\, x' Hx$$

subject to $Ax \geqslant b$, $x \geqslant 0$

where $h = Ec - \sum_{i=1}^{m_1} p_i E(T_i) - g'(EM)$, and Q^i and H are positive semi-definite symmetric matrices for $i = 1, 2 \cdots m_1$. The term $(x'Q^i x)^{\frac{1}{2}}$ with non-negative values is called the standard error term. The problem (a), (b), (c) and (d) will be called proper if they possess the finite optimum. Necessary and/or sufficient conditions for the programs (a), (b), (c) and (d) to be proper are obtained for $x \in R^n_+$.

It is indicated there that the programs of the type (4.7) are difficult to solve directly, but can be solved via dual programs. The programs (4.6) and (4.7) come out to be convex, but non-differentiable somewhere in the region of consideration. By duality theory, the dual problem for (4.7) is the following:

$$\begin{aligned}
&\underset{y \geq 0}{\text{Max:}} \quad b'y - \tfrac{1}{2}\, u'Su \\
&\text{subject to} \quad A'y - \sum_{i=1}^{m_1} Q^i w_i - Su \leq h \\
&\qquad\qquad w_i' Q^i w_i \leq 1, \ i=1,2\text{-----}m_1 ,
\end{aligned}$$

which is a concave and differentiable quadratic program. This program, then, can be solved by the available methods in mathematical programming which in turn, will give the optimal solution for the primal (4.7). The duality theory has been established with the help of Dorn [37] , Eisenberg [49] .

Chapter V is divided into two parts. The saddle point theorem and optimality conditions are obtained for the two stage program and the second stage program. The problem which is considered in the part I is the following:

$$\begin{aligned}
&\underset{x}{\text{Minimize}} \quad E\left[\phi(c,x) + \min_{y} \{\psi(d,y)\}\right] \\
&\text{subject to} \quad f(x) \geq 0 \\
&\qquad\qquad g(x) + h(y) \geq b, \text{ almost all } b,
\end{aligned}$$

where ϕ, ψ and each component of the vector valued functions $-f$, $-g$ and $-h$ are convex functions of their respective arguments. c,d and b are random vectors having random elements with known distributions.

It is observed that the set $K = K_1 \cap K_2$ of feasible solutions is convex, where

$$K_1 = \{x / f(x) \geq 0\} \text{ and } K_2 = \{x / \exists y \text{ such that } g(x) + h(y) \geq b, \text{ almost all } b\}$$

The decision equivalent deterministic program is obtained as follows:

$$\underset{x \in K}{\text{Min}} \left[\Phi(x) + Q(x) \right]$$

which is convex and continuous in the interior of K. Consider the second stage program for given b and x

$$\text{Min} \quad \Psi(d,y)$$

$$\text{subject to} \quad h(y) \geq b - g(x).$$

The Lagrangian for this problem is denoted by

$$\nu(y, \pi(b,x)) = \Psi(d,y) + \langle \pi(b,x), b - g(x) - h(y) \rangle$$

where $\pi(b,x) \in V = \{\pi(b,x) / \ \pi(b,x) \geq 0, \ \nu(y, \pi(b,x))$ attains its minimum$\}$. Saddle point theorem and some optimality criteria are developed for this program. With the help of these results optimality conditions and saddle point theorem are obtained for the two-stage program. It is indicated that all the results obtained above will hold for the nonconvex functions also, because, in the proof the convexity of the functions is not taken into account .

In the second part of this chapter the following problem is considered:

$$\underset{x \geq 0}{\text{Inf}} \ E \left[\Phi(c,x) + \underset{y \geq 0}{\text{Inf}} \{d'y\} + \underset{z}{\text{Inf}} \{\Psi(z)\} \right]$$

$$\text{subject to} \quad Ax \geq b$$

$$P\left[Tx + Wy \geq p\right] \geq \alpha$$

$$g(x) + h(y) \geq q, \text{ almost all } q,$$

where Φ, Ψ , g and h are as defined above. P, E, α , are as defined earlier. It is assumed that the random variables in c,T,W,q and p are known completely by their distributions. The random elements in T and p are assumed to be normally distributed with known means and variances.

As shown in the first part, the set of feasible solutions for this problem also is convex. The decision equivalent deterministic program has been obtained in the following form.

$$\text{Inf:} \quad c'x + \sum_{i=1}^{m_1} (x'Q^i x)^{\frac{1}{2}} + f(x)$$

$$\text{subject to} \quad Ax \geqslant b, \; x \geqslant 0,$$

where Q^i is a positive semi-definite symmetric matrix. This program is shown to be convex which may not be differentiable. Under the condition that the function f is continuously differentiable, the duality theorem has been established. The dual problem comes out to be

$$\sup_{v \geqslant 0} \; b'v + f(u) - u' \nabla f(u)$$

$$\text{subject to} \quad A'v - \nabla f(u) - \sum_{i=1}^{m_1} Q^i u_i \leq c$$

$$u_i' Q^i u_i \leq 1, \; i=1,2,\text{----}m_1 \, ,$$

Chapter VI is also divided into two parts. In this chapter the results established are for the deterministic programs obtained from the generalization of the deterministic programs of Chapter 4 and Chapter 5. Part I consists of the establishment of the duality theorem for the following general symmetric dual programs:

$$\text{(Primal)} \qquad \max_{x \geqslant 0} \; h'x - \tfrac{1}{2} x' Sx - \tfrac{1}{2} y'Ty - \sum_{i=1}^{r} (x'Q^i x)^{\frac{1}{2}}$$

$$\text{subject to} \quad Ax \leq b + Ty + \sum_{j=1}^{t} P^j y_j$$

$$y_j' P^j y_j \leq 1, \; j=1,2,\text{----}t \qquad \text{and}$$

(Dual) $\underset{w \geqslant 0}{\text{Min.}}\ b'w + \frac{1}{2} w'Tw + \frac{1}{2} u'Su + \sum_{j=1}^{t} (w'P^j w)^{\frac{1}{2}}$

subject to $A'w + Su + \sum_{i=1}^{r} Q^i u_i \geqslant h$

$u_i' Q^i u_i \leq 1,\ i = 1,2,\text{---}r$

where A is mxn matrix and S, Q^i(i=1,2---r) are nxn, T, P^j(j=1,2---t) are mxm non-negative definite symmetric matrices. Symmetric duality and self-duality is established for these programs.

In the second part the following programs are considered:

$\underset{x \geqslant 0}{\text{Min.}}\ f(x) + c'x + \sum_{i=1}^{r} (x'Q^i x)^{\frac{1}{2}} + y' \nabla g(y) - g(y)$

subject to $Ax + \nabla g(y) + \sum_{j=1}^{t} P^j y_j \geqslant b,$

$y_j' P^j y_j \leq 1,\ j=1,2\text{---}t,$ and

$\underset{v \geqslant 0}{\text{Max.}}\ f(u) - u' \nabla f(u) - g(v) - \sum_{j=1}^{t} (v'P^j v)^{\frac{1}{2}} + b'v$

subject to $A'v \leq c + \nabla f(u) + \sum_{i=1}^{r} Q^i u_i$

$u_i' Q^i u_i \leq 1,\ i=1,2\text{---}r$

Symmetric duality and self-duality for these problems are developed.

CHAPTER - II

BOUNDS FOR THE STOCHASTIC NONLINEAR PROGRAMMING PROBLEMS*

1. INTRODUCTION: In many optimization problems arising in industry economics etc. the parameters involved in the objective functions as well as in constraints may be known to the organisation to the extent of knowing them as random variables with specified probability distributions. Also, there exist situations where some of the random variables may depend upon the other random variables, such as the requirement or demand vector or availabilities (supplies received from different sources) may be function of random costs at the fluctuating market and vice versa. Under such circumstances it is desired to find the expected maximum profit.

---

* This chapter isbased on the author's published paper [117].

We begin by formulating the problem mathematically in the suitable form. Though it is very difficult to solve such types of problems specifically when the random variables are continuously distributed, in which case they may necessitate an infinite number of mathematical programming problems. But there are some related problems quite important in their own right which can be easily solved with the help of available mathematical programming techniques.

We investigate some properties of the problem considered when the random variables are treated just as parameters. It is shown that the nature of the value of the objective function after the decision has been taken depends upon the nature of the objective function of the original problem. This property helps us to investigate some results for the stochastic formulation of the problem.

We then consider the stochastic case of the problem. The purpose here, is to find some bounds for the expected maximum profit. One upper bound and three lower bounds are obtained for the various cases of the problem. These bounds are of great help in finding out the good estimate of the objective function. All the results obtained are supported by numerical examples.

The results given in this chapter may be considered as generalizations of those of Mangasarian [79] . In addition it contains some results which are not included in [79] . Another difference is that while Mangasarian has dealt with E^n only we have often dealt with more general abstract vector spaces.

2. MATHEMATICAL FORMULATION AND DESCRIPTION OF THE PROBLEM

A class of stochastic non-linear programming problems can be mathematically represented as:

$$\text{Maximize} \quad \Phi(c,x)$$
$$(2.1) \quad \text{subject to} \quad g(a,x) \leq b$$

where Φ is a scalar function of the k-dim vector c and the n-dim. vector x, g is an m-dim vector each component of which is a scalar function of r-dim vector a_i, i=1,2----m, and n-dim vector x and b is an m-dim vector. All the matrices c, a and b contain elements which are random variables in R (the set of reals) with known distribution functions.

We assume that the probability space $(\Omega, \mathcal{F}, P_r)$ is given, in which Ω is a Borel subset of R^N, $N = k + mr + m$, $\mathcal{F}$ is a σ-field on Ω which includes the Borel sets, P_r is the probability measure defined on $\mathcal{F}$ and $\mathcal{F}$ is completed with respect to P_r. Let us assume that the coordinates of a point belonging to Ω can be associated with the components of the collection of three matrices c, a and b of dimensions 1 x k, m x n and 1 x m respectively. Thus, c, a and b can be written as functions of the random variable $w \in \Omega$ (specifically projections Ω_c, Ω_a and Ω_b, where $\Omega = \Omega_c \times \Omega_a \times \Omega_b$) i.e. as c(w), a(w) and b(w) respectively.

The stochastic programming problem (2.1), thus can be, expressed as follows:

$$\text{Maximize} \quad \Phi(c(w),x)$$
$$(2.2) \quad \text{subject to} \quad g(a(w),x) \leq b(w).$$

Because of the randomness prevailed in the problem, maximum may not exist, so we shall write 'find the sup of' in place of 'maximize'.

Now we discuss about a class of problems which will come into the picture later on.

Consider the following sets of feasible solutions:

(2.3) $$B_{p(w)} = \left\{ x / \underset{x \in X}{g(a(w),x) \leq b(w)} \right\}$$

(2.4) $$B_{Ep(w)} = \left\{ x / \underset{x \in X}{g(Ea(w),x) \leq E\,(b(w))} \right\}$$ and

(2.4a) $$B_{p_E(w)} = \left\{ x / \underset{x \in X}{E(g(a(w),x) - b(w)) \leq 0} \right\}$$

where $p(w) = (c_1(w), \text{———} c_k(w),\ a_{11}(w), \text{———} a_{mr}(w),\ b_1(w), \text{———} b_m(w))$

and E denotes the expectation taken with respect to $p(w)$ and X is the real vector space.

Then, consider the following programming problems:

(2.5) $$E \sup_{x \in B_{p(w)}} \phi(c(w),x)$$

(2.6) $$\sup_{x \in B_{Ep(w)}} E\,\phi(c(w),x)$$

(2.7) $$E_a \left\{ \sup_{x \in B_{Ep(w)}} E\,\phi(c(w),x) \text{ for fixed } a(w) \right\}$$

(2.8) $$E_{ab} \left\{ \sup_{x \in B_{E_c p(w)}} E\,\phi(c(w),x) \right\}$$

(2.9) $$E\,\phi(c(w),x_E)$$

where x_E is the solution of the problem

(2.10) $$\sup_{x \in B_{Ep(w)}} \phi(Ec(w),x)$$

(2.11) $$E_a \left\{ \sup_{x \in B_{Ep(w)}} \phi(Ec(w),x) \text{ for fixed } a(w) \right\}$$

and

(2.12) $$E_{ab} \left\{ \sup_{x \in B_{E_c p(w)}} \phi(Ec(w),x) \right\}$$

where E_a, E_{ab} and E_c denote the expectations taken with respect to a, a & b and c respectively.

All these problems (2.5) through (2.12) are considered in later stages where bounds are obtained for the expected maximum profit. These problems except a few of them are extremely difficult to solve specially when c(w), a(w) and b(w) are continuously distributed and in that case they necessitate an infinite number of programming problems.

Problem (2.5) is that of finding the expected value of the supremum of the objective function and is known as the solution of the 'wait and see' problem in the literature. The solution of (2.5) may give the good prediction for the 'wait and see' problem because many business decisions are made in such a fashion that one, first waits for random c, a and b respectively and then selects x optimally. The problem (2.5) does not give the specific value of x though it provides the expected supremum of the profit function.

The solution of the problem (2.6) gives the supremum of the expected value of the objective function. Problems (2.7) and (2.8) are similar to the problem (2.6). To solve the problem (2.6), it is not much difficult, since it becomes a non-stochastic problem. Problems (2.6), (2.7) and (2.8) are important in those situations when there is no time to observe the random variables but have to make decisions before-hand. Such a solution

is called 'here and now' solution. In these problems, having the knowledge of only probability distributions of random variables the optimal decisions are made without observing them. These problems provide us the immediate values of x's giving the supremum value, but this is not the case in problem (2.5), though the solution of (2.5) is more favourable than those of (2.6), (2.7) and (2.8). This is because an estimated solution is less reliable than the solution obtained after observing the random variables. In different situations, the problems (2.5),(2.6),(2.7) and (2.8) have frequent applications.

Problems (2.10), (2.11) and (2.12) are less important than those of (2.5), (2.6), (2.7) and (2.8), since the problems (2.10), (2.11) and (2.12) are solved when the random variables are not observed but are replaced by their expected values knowing their distributions only. These problems are also not difficult to solve by the available techniques of mathematical programming.

3. PARAMETRIC CONSIDERATION OF THE PROBLEM

Let p be the vector $(c_1, \text{---} c_k, a_{11}, \text{---} a_{mr}, b_1, \text{---} b_m)$. We refer to $W = \{p\}$ as a parameter space and to p as the state of the nature.

Let X and $\overline{W}$ be real vector spaces with abelian law group denoted by +, viz., $X \times X \ni (x_1, x_2) \rightarrow x_1 + x_2 \in X$ (respectively $\overline{W} \times \overline{W} \ni (w_1, w_2) \rightarrow w_1 + w_2 \in \overline{W}$) and its external composition law, multiplicatively viz. $R \times X \ni (\lambda, x) \rightarrow \lambda x \in X$ (respectively $R \times \overline{W} \ni (\lambda, w) \rightarrow \lambda w \in \overline{W}$). Let W be a convex subset of $\overline{W}$ where $\overline{W} = \overline{W}_1 \times \overline{W}_2 \times \overline{W}_3$. Then, for the projections of W, one can write $W_1 = W_c$, $W_2 = W_a$, $W_3 = W_b$ respectively.

Now we come to the definition of convex and quasi-convex functions. Let U be a lattice with the order relation $<$. The supremum for any two points u_1 and u_2 in U, is defined by $u_1 \vee u_2$.

<u>Definition</u>: A function $f: X \to U$ is said to be convex if for all $\lambda \in [0,1]$ and for all $x_1 \in X$ and $x_2 \in X$,

$$f(\lambda x_1 + (1-\lambda)x_2) < \lambda f(x_1) + (1-\lambda) f(x_2).$$

<u>Lemma 1.</u>

The mapping $f: X \to U$ is convex if and only if the supergraph

$$A_u = \{(x,u) / x \in X,\ u \in U,\ f(x) < u\}$$

is convex.

<u>Proof</u>. Suppose that f is convex. Then, for all $\lambda \in [0,1]$ and for all $x_1 \in X$ and $x_2 \in X$ and for all $u_1 \in U$ and $u_2 \in U$ such that $f(x_1) < u_1$ and $f(x_2) < u_2$, we see that $(x_1, u_1) \in A_{u_1}$ and $(x_2, u_2) \in A_{u_2}$.

By assumption

$$f(\lambda x_1 + (1-\lambda)x_2) < \lambda f(x_1) + (1-\lambda) f(x_2)$$

$$< \lambda u_1 + (1-\lambda) u_2.$$

This implies that

$$(\lambda x_1 + (1-\lambda)x_2,\ \lambda u_1 + (1-\lambda)u_2) \in A_{\lambda u_1 + (1-\lambda) u_2}$$

i.e. A_u is convex.

To prove the other part, assume that A_u is convex. Since, the points $(x_1, f(x_1)) \in A_{f(x_1)}$ and $(x_2, f(x_2)) \in A_{f(x_2)}$ we have, for all $\lambda \in [0,1]$,

$$(\lambda x_1 + (1-\lambda)x_2,\ \lambda f(x_1) + (1-\lambda)f(x_2)) \in A_{\lambda f(x_1)+(1-\lambda)f(x_2)}$$

This means that

$$f(\lambda x_1 + (1-\lambda)x_2) \angle \lambda f(x_1) + (1-\lambda) f(x_2).$$

Hence, the lemma is proved.

Definition: A mapping $f: X \longrightarrow U$ is said to be quasi-convex if for all $x_1 \in X$ and $x_2 \in X$ and for all $\lambda \in [0,1]$,

$$f(\lambda x_1 + (1-\lambda)x_2) \angle f(x_1) \triangledown f(x_2).$$

Lemma 2. A mapping $f: x \longrightarrow U$ is quasi-convex if and only if the supergraph

$$A_u = \{(x,u)/x \in X,\ u \in U,\ f(x) < u\}$$

is convex.

Proof. Let $f: X \longrightarrow U$ be quasi-convex. Then, for all $\lambda \in [0,1]$, and for all $x_1 \in X$ and $x_2 \in X$,

(3.1) $\quad f(\lambda x_1 + (1-\lambda)x_2) \angle f(x_1) \triangledown f(x_2) \angle u_1 < u_2$ (say)

where $u_1 \in U$ and $u_2 \in U$. Then, the right-hand side inequalities imply that

$$f(x_1) \angle u_1$$

and $\quad f(x_2) \angle u_2.$

It follows that $(x_1,u_1) \in A_{u_1}$ and $(x_2,u_2) \in A_{u_2}$. Also, by the right-hand-side inequalities of (3.1) we have

(3.2) $f(x_1) \vee f(x_2) < \lambda u_1+(1-\lambda)u_2$.

Then, from (3.1) and (3.2) it follows that

$$(\lambda x_1+(1-\lambda)x_2,\ \lambda u_1+(1-\lambda)u_2) \in A_{\lambda u_1+(1-\lambda)u_2}$$

i.e. A_u is convex.

Conversely, let A_u be convex. This, implies that $A_{f(x_1)\vee f(x_2)}$ is convex, where $x_1 \in X$ and $x_2 \in X$.

Now $(x_1,\ f(x_1)\vee f(x_2)) \in A_{f(x_1)\vee f(x_2)}$

and $(x_2,\ f(x_1)\vee f(x_2)) \in A_{f(x_1)\vee f(x_2)}$.

Then, for all $\lambda \in [0,1]$, we have

$$(\lambda x_1+(1-\lambda)x_2,\ f(x_1)\vee f(x_2)) \in A_{f(x_1)\vee f(x_2)}$$

This implies that $f(\lambda x_1+(1-\lambda)x_2) < f(x_1)\vee f(x_2)$,

i.e. f is quasi-convex.

Hence the result is proved.

Let $\mathcal{P}(X)$ be the collection of all subsets of X, and let the family $\{B_p\}_{p\in W} \subset \mathcal{P}(X)$, where $B_p = \{x \in X/p \in W\}$. We assume that for all $\lambda \in [0,1]$, and all $x_1 \in B_{p_1}$ and $x_2 \in B_{p_2}$

(3.3) $\lambda x_1+(1-\lambda)x_2 \in B_{\lambda p_1+(1-\lambda)p_2}$.

By this assumption we see that the set B_p is convex. It is assumed that (3.3) is satisfied throughout the chapter.

One can easily see that under the hypothesis (3.3) the family $\{B_p\}_{p\in W}$, for fixed a, is a convex subset of X.

Let U be a Riesz space. We denote by Q(X,U) the set of quasi-convex functions from X to U and by K(X,U) the set of convex functions from X to U. Also, we denote by K(W x (X), U) the set of functions from W x X to U which are convex on W only and by K(W x X, U) the set of functions from W x X to U which are convex on W and X both. Similar notation denote for the sets of quasi-convex functions.

Theorem 1. Let $\Phi \in K(W_1 x(X), R)$ and let $\alpha : W \rightarrow R$ be defined by $\alpha(p) = \sup_{x\in B_p} \Phi(c,x)$, then $\alpha \in K(W_1 x(W_2 x W_2), R)$.

Proof. For all $\lambda \in [0,1]$, we assume that

$$\alpha(\lambda p_1+(1-\lambda)p_2) = \sup_{x\in B_{\lambda p_1+(1-\lambda)p_2}} (\lambda c_1+(1-\lambda)c_2,x)$$

$$= \Phi(\lambda c_1+(1-\lambda)c_2,x^*).$$

Then, for all $\in > 0$ and for all $\lambda \in [0,1]$, there exist $p_1 \in W$ and $p_2 \in W$ such that

$$\sup_{x\in B_{p_1}} \Phi(c_1,x) > \Phi(c_1,x^*) - \in$$

and

$$\sup_{x\in B_{p_2}} \Phi(c_2,x) > \Phi(c_2,x^*) - \in .$$

Then, we see that

$$\lambda \sup_{x\in B_{p_1}} \phi(c_1,x) + (1-\lambda) \sup_{x\in B_{p_2}} \phi(c_2,x)$$

$$> \lambda\phi(c_1,x^*)+(1-\lambda)\,\phi(c_2,x^*) - \in$$

$$> \phi(\lambda c_1+(1-\lambda)c_2,x^*) - \in$$

$$= \alpha(\lambda p_1+(1-\lambda)p_2) - \in .$$

Since $\in$ is arbitrary, we have

$$\lambda\alpha(p_1)+(1-\lambda)\alpha(p_2) > \alpha(\lambda p_1+(1-\lambda)p_2) \quad .$$

Remark 1. If $\phi\in K(W_1 x\, X, R)$ $(\phi\in K((W_1)x\, X, R))$

then $\alpha \in K(W, R)$ $(\alpha\in K(W_1)x\, W_2 x\, W_3, R))$.

Similar results can be established for concave functions.

Theorem 2. Let $\phi\in Q(W_1 x(X), R)$ and let α be as defined in Theorem 1, then $\alpha\in Q(W_1 x(W_2 x W_3), R)$.

Proof. For all $\in > 0$, $\exists\, x_1 \in B_{p_1}$ and $x_2 \in B_{p_2}$ not necessarily distinct such that

$$\sup_{x\in B_{p_1}} \phi(c_1,x) < \alpha(p_1) + \in$$

and

$$\sup_{x\in B_{p_2}} \phi(c_2,x) < \alpha(p_2) + \in .$$

Then, we have

$$\alpha(\lambda p_1+(1-\lambda)p_2) \leq \sup_{x\in B_{\lambda p_1+(1-\lambda)p_2}} \phi(\lambda c_1+(1-\lambda)c_2,x)$$

$$= \phi(\lambda c_1+(1-\lambda)c_2, x^*)$$

$$\leq \phi(c_1,x^*) \vee \phi(c_2,x^*)$$

$$\leq \sup_{x\in B_{p_1}} \phi(c_1,x) \vee \sup_{x\in B_{p_2}} \in(c_2,x)$$

$$\leq \alpha(p_1) \vee \alpha(p_2) + \in .$$

Since $\in$ is arbitrary, we have the required result.

Remark 2. If $\phi\in Q(W_1 x X,R)$ (or $\phi\in Q((W_1) x X,R)$)

then $\alpha\in Q(W,R)$ ($\alpha\in Q((W_1) x W_2 x W_3, R)$).

4. BOUNDS FOR THE STOCHASTIC CASE

Let W be an open parallelopiped of R^N, $N = k + mr + m$, where k, m, and r are all positive integers. p is a random variable defined on the probability space $(\Omega, \mathcal{F}, P_r)$ with range W, i.e. $p:(\Omega, \mathcal{F}, P_r) \rightarrow W$, such that if μ is the probability law of p, then

$$P_r(p^{-1}(A)) = \mu(A) \text{ , where } A\subset W$$

and

$$P_r(\Omega) = \mu(W) = 1 .$$

Let $W = W_c x W_a x W_b$, where W_c, W_a and W_b are projections of W.

The mathematical expectation of p is given by

$$E\, p(w) = \int_{\Omega} p(w)\, dP_r(w) = \int_{W} \beta\, d\mu(\beta)$$

which is assumed to be finite. Here $p(w) = \beta$, that is for $w \in \Omega$, $p(w) = \beta \in W$. We assume that

$$\bigcap_{p \in W} B_{p(w)} \text{ is not null.}$$

<u>Theorem 3.</u> Let $p: (\Omega, \mathcal{F}, P_r) \to W$. Then

(i) $$E \sup_{x \in B_{p(w)}} \phi(c(w),x) \geqslant \sup_{x \in B_{E_{p(w)}}} E\, \phi(c(w),x) \geqslant E\, \phi(c(w),x_E)$$

$$\geqslant \sup_{x \in B_{E\, p(w)}} \phi(Ec(w),x)$$

$$= \phi(Ec(w),x_E)$$

(ii) $$E \sup_{x \in B_{p(w)}} \phi(c(w),x) \geqslant E \sup_{x \in B_{E_c p(w)}} E\phi(c(w),x) \geqslant E\, \phi(c(w),x_{E_c})$$

$$\geqslant E \sup_{x \in B_{E_c p(w)}} \phi(Ec(w),x)$$

where only for the last inequality in (i) and (ii), it is assumed that $\phi \in K(W_1 x(X),R)$.

<u>Proof.</u> We shall prove the second set of inequalities. The proof of the first set of inequalities will follow immediately.

By theorem 1, $\alpha(p) = \sup_{x \in B_{p(w)}} \phi(c(w),x)$ and $\alpha \in K(W_1 x(W_2 x\, W_3),R)$.

$\sup_{x \in B_{p(w)}} \phi(c(w),x)$ is a random variable defined on the probability space $(\Omega, \mathcal{F}, P_r)$ for fixed $a(w)$ and $b(w)$.

Thus,

$$E \sup_{x \in B_{p(w)}} \phi(c(w),x) = \int_{\Omega} \sup_{x \in B_{p(w)}} \phi(c(w),x) \, dP_x(w)$$

has meaning.

Since

$$\sup_{x \in B_{p(w)}} \phi(c(w),x) \geq \phi(c(w),x) \text{ for all } x \in B_{p(w)}$$

Thus,

$$E_c \sup_{x \in B_{p(w)}} \phi(c(w),x) \geq E_c \phi(c(w),x) \quad \text{for all } x \in B_{p(w)} .$$

This implies that

$$E_c \sup_{x \in B_{p(w)}} \phi(c(w),x) \geq \sup_{x \in B_{E_c p(w)}} E_c \phi(c(w),x)$$

Since $B_{E_c p(w)} = B_{p(w)}$.

Hence,

$$E \sup_{x \in B_{p(w)}} \phi(c(w),x) \geq E \sup_{x \in B_{E_c p(w)}} E_c \phi(c(w),x) .$$

For the second inequality since we have

$$\sup_{x \in B_{E_c p(w)}} E_c \phi(c(w),x) \geq E_c \phi(c(w),x) \quad \text{for all } x \in B_{E_c p(w)}$$

which implies that

$$\sup_{x \in B_{E_c p(w)}} E \phi(c(w),x) \geq E_c \phi(c(w), x_{E_c})$$

where $x_{E_c} \in B_{E_c p(w)}$ and is the solution of the problem

$$\sup_{x \in B_{E_c p(w)}} \phi(Ec(w),x) .$$

Utilizing the Jensen's inequality for the function $\phi(c(w), x_{E_c})$ we obtain the third inequality and thus the result is proved.

Remark 3. If $\phi \in K(W_1 \times X, R)$, the inequalities in the above theorem are also valid. In this case we can replace the feasible set $B_{Ep(w)}$ for the second term in the two sets of inequalities above by the set $B_{\underline{p}(w)}$ without affecting the result.

Remark 4. If the random variables b's are convex functions of the random variables c's, the inequality sets (i) and (ii) of Theorem 3 remains unaffected, and if c's are convex functions of b's then, still the theorem holds with E_c replaced by E_b.

Theorem 4. Let a(w) and b(w) be fixed. Let the family $\{B_{p(w)}\}_{p(w) \in W}$ be partially ordered by inclusion and monotone i.e. $B_{p'(w)} \subset B_{p''(w)}$ if $p'(w) \leq p''(w)$, where p(w) is also partially ordered. Let the probability law μ of p(w) has a compact support G in R^N. Then, there exists a value $p(w_s)$ of a random variable p(w) such that,

$$(4.1) \quad E_{ab} \sup_{x \in B_{p(w_s)}} \phi(c(w_s), x) \geqslant E_{ab} \sup_{x \in B_{E_c p(w)}} E_c \phi(c(w), x).$$

The solution of this problem is called the 'fat' solution. The proof is similar to the given in [5].

Remark 5. If $\phi \in K(W_1 \times X, R)$, the result (4.1) holds good for fixed a and E_{ab} replaced by E_a in the left-hand side.

Now we give a theorem of Madansky [75] for our case.

Theorem 5. Let a(w) and b(w) be fixed. Let $\phi(c(w), x) \in K(W_1 \times (X), R)$ and be a continuous function on W_1. If the probability law μ of p(w) has a

bounded support, that, if $\{-\infty < p_1(w) < p(w) < p_2(w) < +\infty\}$ and if $\alpha(p(w))$ has no discontinuity over the boundary, then

$$(4.2)\quad E_{ab} \sum_{i=1}^{2^k} \left\{ \prod_{j=1}^{k} (-1)^{t_{ij}} \left[\frac{p(w)_{t_{ij},j} - (Ep(w))_j}{p(w)_{2j} - p(w)_{1j}}\right] \cdot \alpha(p(w)_{3-t_{i1}}, \text{----} p(w)_{3-t_{ik}}, p) \right\}$$

$$\geqslant E \sup_{x \in B_{p(w)}} \phi(c(w), x)$$

where t_i is a k-dimensional vector $(t_{i1}, \text{----} t_{ik})'$ whose components are 1 and/or 2's and the set of 2^k vectors t_i, $i = 1, 2, \text{----} 2^k$ is the set of all possible arrangements of 1's and/or 2's taken k at a time and $(Ep(w))_j$ is the expected value of the jth component of $p(w)$.

Remark 6. If $\phi \in K(W_1 x X, R)$, then k is replaced by $k + m$ and E_{ab} by E_a in (4.2).

Theorem 6. Let the assumptions of Theorem 4 be satisfied for fixed $a(w)$. Let $\phi \in K(W_1 x X, R)$. Then there exist values $p(w')$ and $p(w'')$ of $p(w)$ such that, for fixed $a(w)$

$$(4.3)\quad E_a \sup_{x \in B_{p(w')}} \phi(c(w'), x) \geqslant E_a \sup_{x \in B_{p(w)}} \phi(c(w), x)$$

$$\geqslant E_a \sup_{x \in B_{p(w'')}} \phi(c(w''), x') \quad .$$

Proof. Let Λ be a compact subset of E^N contained in W. Then, there exists a smallest parallelopiped

$$\{-\infty < \overline{p(w)} < p(w) < p^+(w) < +\infty\}$$

containing $\wedge$ and contained in W. We denote this parallelopiped by

$$\begin{pmatrix} \overline{c(w)} \\ \overline{a(w)} \\ \overline{b(w)} \end{pmatrix} \leq \begin{pmatrix} c(w) \\ a(w) \\ b(w) \end{pmatrix} \leq \begin{pmatrix} c^+(w) \\ a^+(w) \\ b^+(w) \end{pmatrix} .$$

Taking a(w) fixed, let, in particular,

$$p(w'') = \begin{pmatrix} \overline{c(w)} \\ a(w) \\ \overline{b(w)} \end{pmatrix} \quad \text{and} \quad p(w') = \begin{pmatrix} c^+(w) \\ a\,(w) \\ b^+(w) \end{pmatrix} .$$

Then,

$$x \in B_{p(w'')} \equiv \{ x/g(a(w),x) \leq b(w'') \ , \ x \in X \}$$
$$\equiv \{ x/g(a(w),x) \leq \overline{b(w)} \ , \ x \in X \}$$
$$\Rightarrow \quad x \in B_{p(w)} \equiv \{ x/g(a(w),x) \leq b(w) \ , \ x \in X \}$$
$$\Rightarrow \quad x \in B_{p(w')} \equiv \{ x/g(a(w),x) \leq b(w') \ , \ x \in X \}$$
$$\equiv \{ x/g(a(w),x) \leq b^+(w) \ , \ x \in X \} .$$

This shows that

$$B_{p(w')} \supset B_{p(w)} \supset B_{p(w'')}$$

and, then, we obtain that

$$E_a \sup_{x \in B_{p(w')}} \phi(c(w'),x) \geq E_a \sup_{x \in B_{p(w)}} \phi(c(w),x)$$
$$\geq E_a \sup_{x \in B_{p(w'')}} \phi(c(w''),x) .$$

<u>Remark 7</u>. Let $\phi(c(w),x) \in K(W_1 x(X),R)$. Then, the theorem 6 holds good with fixed a(w) and b(w) and E_a replaced by E_{ab}. If, however, each component of g(a(w),x) belongs to the set $K(W_2 x\ X,\ R)$, we can remove the

condition of fixing $a(w)$. And, then under the assumptions of Theorem 6, we can s-elect $p(w') = (c^+(w), a(w), b^+(w))$ and $p(w'') = (c^-(w), a^+(w), b^-(w))$ satisfying the results (4.3) with E_a removed.

<u>Theorem 7</u>. Let $\phi(c(w),x)$ belong to the set of concave functions from $W_1 x X$ to R which are concave in W_1. Let $Ec(w) > 0$.
Then, for fixed $a(w)$ and $b(w)$, we have

$$E_{ab} \sup_{x \in B_{E_c p(w)}} \phi(Ec(w),x) \geq E \sup_{x \in B_{p(w)}} \phi(c(w),x) \text{----}$$

$$\geq E_{ab} \sup_{x \in B_{E_c p(w)}} E_c \phi(c(w),x) \geq E \phi(c(w), x_{E_c})$$

where x_{E_c} is the solution of $\sup_{x \in B_{E_c p(w)}} \phi(Ec(w),x)$.

<u>Proof</u>. Since, the two sets $B_{p(w)}$ and $B_{E_c p(w)}$ are same. Let $\bar{x}$ be the solution of the problem $\sup_{x \in B_{p(w)}} \phi(c(w),x)$.

Then,

$$\phi(c(w),\bar{x}) = \sup_{x \in B_{p(w)}} \phi(c(w),x)$$

Now, applying the Jensen's inequality for the concave function $\phi(c(w),x)$ we get

$$\phi(E(c(w), x) \geq E_c \phi(c(w),x) \quad \text{for all } x \in B_{p(w)}.$$

Thus, in particular,

$$\phi(Ec(w),\bar{x}) \geq E_c \phi(c(w),\bar{x})$$
$$= E_c \sup_{x \in B_{p(w)}} \phi(c(w),x).$$

This implies that

$$\sup_{x \in B_{E_c p(w)}} \phi(Ec(w),x) \geqslant E_c \sup_{x \in B_{p(w)}} \phi(c(w),x).$$

Hence, the first inequality is satisfied. The proof of other two inequalities is similar to the proof given in the Theorem 3.

The results of Theorem 3 reduce in particular to the case of Mangasarian [79] when $X = R^n$ and $a(w)$ and $b(w)$ are constants.

5. NUMERICAL EXAMPLES

Example 1. Let us consider the following problem:

$$\text{Maximize} \quad c_1 x_1^2 + c_2^2 \cdot x_2^2$$

$$\text{subject to} \quad a_1 x_1 \leq b_1$$

$$a_2 x_2 \leq b_2$$

$$x_1, x_2 \geqslant 0$$

where c_1, c_2, a_1, a_2 and b_1, b_2 are uniformly distributed as follows

$$c_1 \to \left[-\tfrac{1}{2}, \tfrac{1}{2}\right], \quad c_2 \to [0,1], \quad a_1 \to \left[\tfrac{1}{4}, \tfrac{1}{2}\right],$$

$$a_2 \to \left[\frac{1}{5}, \frac{1}{3}\right], \quad b_1 \to [4, 12] \text{ and } b_2 \to [2,10].$$

Then,

$$\max_{x \in B_{p(w)}} \phi(c(w),x) = c_1 \left(\frac{b_1}{a_1}\right)^2 + c_2^2 \left(\frac{b_2}{a_2}\right)^2 \quad \text{for } c_1 \geqslant 0$$

$$= c_2^2 \left(\frac{b_2}{a_2}\right)^2 \quad \text{for } c_1 < 0.$$

Thus,

$$E \max_{x \in B_{p(w)}} \phi(c(w),x) = 276.00 \tag{5.1}$$

Now,

$$E_{ab} \max_{x \in B_{p(ws)}} \phi(c(ws),x) = 897.33 \text{---} > 206.66 \text{----}$$

i.e. fat solution is satisfied.

To verify Theorem 5, we have

$$E_{ab} \left\{ \frac{-\frac{1}{2} - 0}{\frac{1}{2} - (-\frac{1}{2})} \cdot \frac{0 - \frac{1}{2}}{1 - 0} \left[\frac{1}{2}\left(\frac{b_1}{a_1}\right)^2 + \left(\frac{b_2}{a_2}\right)^2 \right] + \frac{\frac{1}{2} - 0}{\frac{1}{2} - (\frac{1}{2})} \quad \frac{1 - \frac{1}{2}}{1 - 0} \left[0 \right] \right.$$

$$\left. - \frac{-\frac{1}{2} - 0}{\frac{1}{2} - (-\frac{1}{2})} \cdot \frac{1 - \frac{1}{2}}{1 - 0} \left[\frac{1}{2} \left(\frac{b_1}{a_1}\right)^2 \right] - \frac{\frac{1}{2} - 0}{\frac{1}{2} - (-\frac{1}{2})} \quad \frac{0 - \frac{1}{2}}{1 - 0} \left[\left(\frac{b_2}{a_2}\right)^2 \right] \right\}$$

$$= 448.666 \text{---}$$

which is clearly greater than 276. A rough estimate for the expected maximum profit could be obtained by averaging 448.666--- and the closed lower bound 206.666---(or 168.5625. This gives 327.666---(or 308.708--) which is off by 18.739 % (or 11.85 %).

To verify Theorem 6, we know that the objective function $\phi \in K(W_1 \times X, R)$ and each component of g belongs to the set $K(W_2 \times X, R)$. Then,

$$\sup_{x \in B_{p(w')}} \phi(c^+(w),x) = 3652 > 675 = \sup_{x \in B_{p(w)}} \phi(c(w),x)$$

at $b_1 = 10, b_2 = 8, a_1 = \frac{1}{3}$, $a_2 = \frac{4}{15}$

and $c_1 = \frac{1}{2}$, $c_2 = \frac{1}{2}$

$$> 0 = \sup_{x \in B_{p(w'')}} \phi(\overline{c(w)},x).$$

<u>Example 2</u>. Now we consider the following problem where b's are convex functions of c's:

$$\text{Maximize} \quad c_1 x_1^2 + c_2^2 x_2^2$$

$$\text{subject to} \quad a_1 x_1 \leq 20\, c_1^2$$

$$a_2 x_2 \leq 30\, c_2$$

$$x_1, x_2 \geq 0,$$

where c_1, c_2, a_1 and a_2 have the same distributions as given in example 1.

Then, we see that

$$E \max_{x \in B_{p(w)}} \phi(c(w), x) = 2706.25$$

$$E_a \max_{x \in B_{E_c p(w)}} \phi(c(w), x) = 1125,$$

$$E\, \phi(c(w), x_{E_c}) = 1125,$$

and

$$E_a \max_{x \in B_{E_c p(w)}} \phi(Ec(w), x) = 843.75.$$

Similarly we shall get all the other results which together with these results satisfy the remarks and theorems given.

<u>Example 3</u>. A case when c's are convex functions of b's.

We consider the following problem:

$$\text{Maximize} \quad \frac{1}{8} b_1 x_1^2 + \frac{1}{576} b_2^2 x_2^2$$

$$\text{subject to} \quad a_1 x_1 \leq b_1$$

$$a_2 x_2 \leq b_2$$

$$x_1, x_2 \geq 0$$

where a's and b's have same distributions as in example 1. We, then, find that

$$E \max_{x \in B_{p(w)}} \phi(b(w),x) = 705.083 \,,$$

$$E_a \max_{x \in B_{E_b p(w)}} \phi(b(w),x) = 550.75 \,,$$

$$E\, \phi(b(w), x_{E_b}) = 550.75 \,,$$

and

$$E_a \max_{x \in B_{E_b p(w)}} \phi(Eb(w),x) = 545.75 \,,$$

which satisfy the inequalities of Theorem 3. Similarly other theorems can be verified easily.

Example 4. Here we take an example where $\phi(c(w),x)$ is a concave function in c and/or x.

We consider the following problem:

$$\text{Maximise} \quad c_1 x_1 - c_1^2 x_1^2 + c_2 x_2 - c_2^2 x_2^2$$

$$\text{subject to} \quad 0 \leq x_1 \leq 3$$

$$0 \leq x_2 \leq 2$$

where c_1 and c_2 are uniformly distributed as

$c_1 \rightarrow [-\frac{1}{2}, 1]$ and $c_2 \rightarrow [0,1]$.

Since

$$\begin{aligned} \max_{x \in B_{p(w)}} \phi(c_1(w),x_1) &= 0 \text{ for } c_1 \leq 0 \\ &= (3c_1 - 9c_1^2) \text{ for } c_1 \in (0, \tfrac{1}{6}] \\ &= \tfrac{1}{4} \text{ for } c_1 \geq \tfrac{1}{6} \,. \end{aligned}$$

where a's and b's have same distributions as in example 1. We, then, find that

$$E \max_{x \in B_{p(w)}} \phi(b(w),x) = 705.083 ,$$

$$E_a \max_{x \in B_{E_b p(w)}} \phi(b(w),x) = 550.75 ,$$

$$E\, \phi(b(w), x_{E_b}) = 550.75 ,$$

and

$$E_a \max_{x \in B_{E_b p(w)}} \phi(Eb(w),x) = 545.75 ,$$

which satisfy the inequalities of Theorem 3. Similarly other theorems can be verified easily.

<u>Example 4</u>. Here we take an example where $\phi(c(w),x)$ is a concave function in c and/or x.

We consider the following problem:

Maximize $\quad c_1x_1 - c_1^2\, x_1^2 + c_2x_2 - c_2^2\, x_2^2$

subject to $\quad 0 \le x_1 \le 3$

$\qquad\qquad\quad 0 \le x_2 \le 2$

where c_1 and c_2 are uniformly distributed as

$c_1 \rightarrow [-\frac{1}{2}, 1]$ and $c_2 \rightarrow [0,1]$.

Since

$$\begin{aligned} \max_{x \in B_{p(w)}} \phi(c_1(w),x_1) &= 0 \text{ for } c_1 \le 0 \\ &= (3c_1 - 9c_1^2) \text{ for } c_1 \in (0, \tfrac{1}{6}] \\ &= \tfrac{1}{4} \text{ for } c_1 \geq \tfrac{1}{6} , \end{aligned}$$

and $\max_{x \in B_{p(w)}} \phi(c_2(w),x) = 2c_2 - 4c_2^2 \quad \text{for } c_2 \in [0, \tfrac{1}{4}]$

$$= \tfrac{1}{4} \text{ for } c_2 \geqslant \tfrac{1}{4} .$$

$$E \max_{x \in B_{p(w)}} \phi(c(w),x) = E \max_{x \in B_{p(w)}} \phi(c_1(w),x) + E \max_{x \in B_{p(w)}} \phi(c_2(w),x)$$

$$= 0.1574 + 0.2291 = 0.3865.$$

Now, $E\,\phi(c(w),x) = \tfrac{1}{4} x_1 - \tfrac{1}{4} x_1^2 + \dfrac{x_2}{2} - \dfrac{x_2^2}{3}$.

$\therefore \max_{x \in B_E p(w)} E\,\phi(c(w),x) = \tfrac{1}{4} = 0.25.$

Now, $\phi(Ec(w),x) = \tfrac{1}{4} x_1 - \tfrac{1}{16} x_1^2 + \tfrac{1}{2} x_2 - \tfrac{1}{4} x_2^2$

$\therefore \max_{x \in B_{Ep(w)}} \phi(Ec(w),x) = \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2} = 0.5$

and $\phi(c(w),x_{E_c}) = 2c_1 - 4c_1^2 + c_2 - c_2^2$,

$\therefore E\,\phi(c(w),x_{E_c}) = -\tfrac{1}{6}$

Hence, we see that

$$0.5 > 0.3865 > 0.25 > -0.166\text{---}$$

i.e. the inequalities of Theorem 7 are satisfied.

CHAPTER - III

DUALITY IN QUADRATIC PROGRAMMING PROBLEMS UNDER UNCERTAINTY

1. INTRODUCTION

The following two-stage quadratic programming under uncertainty is treated in this chapter:

$$(1.1) \quad \underset{x \geqslant 0}{\text{Minimise}} \left[p'x + \tfrac{1}{2}x'Cx + E \underset{y \geqslant 0}{\min} \left\{ q'y + \tfrac{1}{2}\, y'Dy \right\} \right]$$

subject to $Ax + By \geqslant b$, for almost all b,

where b is an $(m\text{x}1)$ random vector with known joint distribution μ and E is the expectation taken with respect to the distribution μ. C and D are $(r\text{x}r)$ and $(s\text{x}s)$ positive semi-definite symmetric matrices A and B are $(m\text{x}r)$ and $(m\text{x}s)$ matrices respectively, p and x are both $(r\text{x}1)$ column vectors and q and y are both $(s\text{x}1)$ column vectors and prime denotes the transpose.

The problem (1.1) is the quadratic version of Dantzig's two-stage linear programming under uncertainty [34] . This problem was termed as 'here and now' problem by Dantzig. The first stage problem is to decide an x and then observe a random b. The second-stage problem consists of minimizing the penalty cost of y(b) subject to the constraints generated by the observed b. The object is to find an x which leads to smallest possible total cost.

A vector x is said to be feasible for the problem (1.1) if it is non-negative and if the second-stage program $\min\limits_{y \geqslant 0} \{q'y + \frac{1}{2}y'Dy\}$ subject to $By \geqslant b - Ax$ is feasible for almost all b. Thus, a vector x is limited to those values for which the constraints are satisfied for almost all b. It is assumed that there exists a non-null closed convex set K of all non-negative x's such that for each $x \in K$ there exists an associated y(b) which is non-negative such that (x,y(b)) is feasible for almost all b. All quantities considered here are assumed to be real.

Let $\mathcal{B}$ be the sample space of all b's. It is assumed that the probability space $(\mathcal{B}, \mathcal{F}_1, \mu)$ is given, where $\mathcal{B}$ is the Borel subset of R^m, $\mathcal{F}_1$ is the σ-field on $\mathcal{B}$ which includes Borel subsets and μ is the probability measure generated by the distribution function also denoted by μ and $\mathcal{F}_1$ is completed with respect to μ .

After reformulating the problem, duality theory and optimality criteria are established by two different techniques using the Hilbert-space methods throughout. Section 3 consists of the derivation of the optimality criteria, the saddle point theorem and the duality theorem with the help of fundamental (Hahn Banach) seperation theorem. All these

results are established in section 4 by the another method corresponding closely to the method of Madansky's paper for the linear version [77]. Some optimality criteria for the second stage program are considered in the last section.

2. REFORMULATION OF THE PROBLEM

Consider the following notation:

$$H = [A, B] \quad , Q = \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}$$

$$c = \begin{pmatrix} p \\ q \end{pmatrix} \quad \text{and} \quad \xi(b) = \begin{pmatrix} x \\ y(b) \end{pmatrix}.$$

Then, the problem (1.1) reduces to the following form:

(2.1) Minimize $\int c' \xi(b) d\mu(b) + \frac{1}{2} \int \xi'(b) Q \xi(b) d\mu(b)$

subject to $H \xi(b) \geqslant b, \quad \xi(b) \geqslant 0$, for almost all b.

In this new version of the problem the decision vector $\xi(b)$ consists of both x and y(b) with x as a function of b, i.e., at first stage one must make the first stage decision x and also specify, in advance, some decision y(b) for the next stage for each possible b. Thus, the two versions of the problem (1.1) and (2.1) are identical.

The corresponding dual of the problem (2.1) can be expressed as follows:

(2.2) Maximize $\int b' \delta(b) d\mu(b) - \frac{1}{2} \int \xi'(b) Q \xi(b) d\mu(b)$

subject to $H' \delta(b) - Q \xi(b) \leq c, \quad \delta(b) \geqslant 0$, for almost all b.

Now, the following restriction of the problem is considered. Let E_x^{r+s} be a subset of vectors of Euclidean (r+s) dimensional space E^{r+s} whose first r-coordinates are the coordinates of the vector x.

Let $\tilde{\mathcal{J}} = \bigcup_{x \in E^r} \; \underset{b \in \mathcal{B}}{X} \; E_x^{r+s}$

where X denotes the Cartesian product. Any element of $\tilde{\mathcal{J}}$ is a collection $[\xi(b)]$ of $\xi(b) \in E_x^{r+s}$. Let $\mathcal{J}$ be the convex subspace of $\tilde{\mathcal{J}}$ such that each member of $\mathcal{J}$ is measurable and square integrable with respect to μ except on the μ-nullsets. It is assumed that, for each member $[\xi(b)]$ of $\mathcal{J}$, $\xi(b)$ is feasible for almost all b. The inner product of any two collections $[\eta(b)]$ and $[\xi(b)]$ of $\mathcal{J}$ is defined as follows:

$$\langle [\eta(b)] \, , \; [\xi(b)] \rangle = \int \eta'(b) \; \xi(b) \, d \, \mu(b).$$

With this inner produce (identifying collections which differ on μ-null sets) $\mathcal{J}$ is a Hilbert space.

Let δ be an element of E^m. Consider the following:

$$\tilde{\mathcal{P}} = \underset{b \in \mathcal{B}}{X} \; E^m \quad .$$

Then, any element of $\tilde{\mathcal{P}}$ is a collection $[\delta(b)]$ of $\delta(b) \in E^m$. Take $\mathcal{P}$ to be the convex sub-space of $\tilde{\mathcal{P}}$ such that each element $[\delta(b)] \in \mathcal{P}$ viewed as a function of b, is measurable and square integrable with respect to μ except on the μ-null sets. By the following inner product defined for any two collections $[\delta(b)]$ and $[\theta(b)]$ of $\mathcal{P}$ as

$$\langle [\delta(b)] \, , \; [\theta(b)] \rangle = \int \delta'(b) \; \theta(b) \, d \, \mu \; (b),$$

(identifying collections which differ on μ-null sets) $\mathcal{P}$ is a Hilbert space.

Let $\tilde{\mathcal{J}} = \bigcup_{x \in E^r} \; \mathop{X}_{b \in \mathcal{B}} E_x^{r+s}$

where X denotes the Cartesian product. Any element of $\tilde{\mathcal{J}}$ is a collection $[\xi(b)]$ of $\xi(b) \in E_x^{r+s}$. Let $\mathcal{J}$ be the convex subspace of $\tilde{\mathcal{J}}$ such that each member of $\mathcal{J}$ is measurable and square integrable with respect to μ except on the μ-nullsets. It is assumed that, for each member $[\xi(b)]$ of $\mathcal{J}$, $\xi(b)$ is feasible for almost all b. The inner product of any two collections $[\eta(b)]$ and $[\xi(b)]$ of $\mathcal{J}$ is defined as follows:

$$\langle [\eta(b)] , [\xi(b)] \rangle = \int \eta'(b) \; \xi(b) \, d \, \mu(b).$$

With this inner produce (identifying collections which differ on μ-null sets) $\mathcal{J}$ is a Hilbert space.

Let δ be an element of E^m. Consider the following:

$$\tilde{\mathcal{O}} = \mathop{X}_{b \in \mathcal{B}} E^m \quad .$$

Then, any element of $\tilde{\mathcal{O}}$ is a collection $[\delta(b)]$ of $\delta(b) \in E^m$. Take $\mathcal{O}$ to be the convex sub-space of $\tilde{\mathcal{O}}$ such that each element $[\delta(b)] \in \mathcal{O}$ viewed as a function of b, is measurable and square integrable with respect to μ except on the μ-null sets. By the following inner product defined for any two collections $[\delta(b)]$ and $[\theta(b)]$ of $\mathcal{O}$ as

$$\langle [\delta(b)] , [\theta(b)] \rangle = \int \delta'(b) \; \theta(b) \, d \, \mu(b),$$

(identifying collections which differ on μ-null sets) $\mathcal{O}$ is a Hilbert space.

Suppose that the squared length of b is measurable and integrable with respect to μ .

Consider the following:

$$[b] = \underset{b \in \mathcal{B}}{X} \; b \in \mathcal{P}, \quad [H\,\xi(b)] = \underset{b \in \mathcal{B}}{X} \; H\,\xi(b) \in \mathcal{P}$$

similarly

$$[c] = \underset{b \in \mathcal{B}}{X} \; c \in \mathcal{T}, \quad [Q\,\xi(b)] = \underset{b \in \mathcal{B}}{X} \; Q\,\xi(b) \in \mathcal{T}$$

and

$$[H'\,\delta(b)] = \underset{b \in \mathcal{B}}{X} \; H'\,\delta(b) \in \mathcal{T}, \quad [0] = \underset{b \in \mathcal{B}}{X} \; 0$$

where $[0]$ is the zero vector of appropriate dimension.

By the notation

$$[H\,\xi(b)] \geqslant [b], \quad [\xi(b)] \geqslant [0], \; [H'\,\delta(b) - Q\,\xi(b)] \leq [c], \; [\delta(b)] \geqslant [0]$$

we mean that

$H\,\xi(b) \geqslant b$, $\xi(b) \geqslant 0$, $H'\,\delta(b) - Q\,\xi(b) \leq c$, $\delta(b) \geqslant 0$ respectively

for almost all b. The problems (2.1) and (2.2), now, reduce to the following form:

(2.3) $\operatorname{Inf} \langle [c], [\xi(b)] \rangle + \frac{1}{2} \langle [\xi(b)], [Q\,\xi(b)] \rangle$

subject to $[H\,\xi(b)] \geqslant [b]$, $[\xi(b)] \geqslant [0]$

$[\xi(b)] \in \mathcal{T}$

and

(2.4) $\sup \langle [b], [\delta(b)] \rangle - \frac{1}{2} \langle [\xi(b)], [Q\,\xi(b)] \rangle$

subject to $[H'\,\delta(b) - Q\,\xi(b)] \leq [c]$

$[\delta(b)] \geqslant [0]$,

$[\delta(b)] \in \mathcal{P}$, $[\xi(b)] \in \mathcal{T}$.

A collection $[\xi(b)]$ ($[\xi(b)]$, $[\delta(b)]$) is said to be primal (dual) feasible if it satisfies the primal (dual) constraints for almost all b.

The two space $\mathcal{T}$ and $\mathcal{D}$ defined above are Hilbert spaces and so is $\mathcal{T} \times \mathcal{D}$. The two problems (2.3) and (2.4), known as the dual programs are, now, defined in Hilbert spaces $\mathcal{T}$ and $\mathcal{T} \times \mathcal{D}$ respectively. H can be viewed as a linear transformation from $\mathcal{T}$ to $\mathcal{D}$ and H' from $\mathcal{D}$ to $\mathcal{T}$ such that

$$\langle [H\,\xi(b)] \;,\; [\delta(b)] \rangle = \langle [\xi(b)] \;,\; [H'\delta(b)] \rangle$$

where $[\xi(b)] \in \mathcal{T}$ and $[\delta(b)] \in \mathcal{D}$. Q can be throught of as a linear, positive semi-definite symmetric operator from $\mathcal{T}$ to $\mathcal{T}$ such that

$$\langle [\xi(b)] \;,\; [Q\,\xi(b)] \rangle = \langle [Q\,\xi(b)] \;,\; [\xi(b)] \rangle$$

so that Q is a self-adjoint operator i.e. $Q = Q^*$. The two collections $[\xi(b)] \in \mathcal{T}$ and $[\eta(b)] \in \mathcal{T}$ are said to be equivalent if $[\xi(b)] = [\eta(b)]$ almost everywhere. From here onward we shall write $[\xi]$ and $[\delta]$ in place of $[\xi(b)]$ and $[\delta(b)]$ respectively unless otherwise specified.

The problem (2.3) is, now, expressed as the supremum problem called primal and denoted by (P)

$$\sup \quad f([\xi]) = -\langle [c] \,,\, [\xi] \rangle - \tfrac{1}{2} \langle [\xi] \,,\, [Q\,\xi] \rangle$$

(2.5) subject to $[H\,\xi] \geq [b]$

(2.6) $\qquad [\xi] \geq [0]\,, \quad [\xi] \in \mathcal{T}$

and the problem (2.4), as infimum problem called dual and denoted by (D)

$$\text{Inf} \quad g([\xi], [\delta]) = -\langle [b] \,,\, [\delta] \rangle + \tfrac{1}{2} \langle [\xi] \,,\, [Q\,\xi] \rangle$$

(2.7) subject to $[H'\delta - Q\,\xi] \leq [c]$,

(2.8) $\qquad [\delta] \geq [0]\,, \quad [\delta] \in \mathcal{D}\,, \quad [\xi] \in \mathcal{T}$.

Let $\mathcal{G}^*$ be the conjugate space of $\mathcal{G}$ i.e. the set of all continuous linear functionals on $\mathcal{G}$. Then $\mathcal{G}^*$ is also a space of bounded linear functionals on $\mathcal{G}$.

The Lagrangian function for the problem (P) can be expressed as follows:

$$\Phi = - \langle [c], [\xi] \rangle - \tfrac{1}{2} \langle [\xi], [Q\xi] \rangle + \delta^*[H\xi - b]$$

where $\delta^* \in \mathcal{G}^*$. Then δ^* is a non-negative continuous linear functional in $\mathcal{G}^*$. By the Reizz-representation theorem for the bounded linear functionals [41], for any bounded linear functional δ^* in $\mathcal{G}^*$, there exists a unique member of $\mathcal{G}$, say $[\lambda]$, such that

$$\delta^*[\delta] = \langle [\delta], [\lambda] \rangle$$

for all $[\delta] \in \mathcal{G}$.

Hence, Lagrangian Φ becomes

$$\Phi = \phi([\xi], [\delta]) = - \langle [c], [\xi] \rangle - \tfrac{1}{2} \langle [\xi], [Q\xi] \rangle + \langle [H\xi], [\delta] \rangle - \langle [b], [\delta] \rangle$$

$$[\xi] \geq [0], \ [\delta] \geq [0], \ [\xi] \in \mathcal{T}, \ [\delta] \in \mathcal{G}.$$

By a similar argument, the Lagrangian expression for the dual problem (D) is

$$-\Phi = \langle [c], [\xi] \rangle + \tfrac{1}{2} \langle [\xi], [Q\xi] \rangle - \langle [H\xi], [\delta] \rangle + \langle [b], [\delta] \rangle$$

$$= \Psi([\xi], [\delta]),$$

3. <u>OPTIMAL DUAL VARIABLES</u>

Weak and strong duality theorems are established in this section for the problems (P) and (D). Optimality criteria are also discussed.

Theorem 3.1. (Weak duality). For all feasible solutions to problems (P) and (D) respectively

(3-0) $\quad f([\xi]) \leq g([\xi], [\delta])$

where $[\xi] \in \mathcal{T}$ and $[\delta] \in \mathcal{P}$

Proof. Let the collections $[\xi] \in \mathcal{T}$ and $([\xi],[\delta]) \in \mathcal{T} \times \mathcal{P}$ be feasible for problems (P) and (D) respectively.

Then,

$$
\begin{aligned}
g([\xi], [\delta]) &= -\langle [b], [\delta] \rangle + \tfrac{1}{2}\langle [\xi], [Q\xi] \rangle \\
&\geqslant -\langle [H\xi], [\delta] \rangle + \tfrac{1}{2}\langle [\xi], [Q\xi] \rangle \\
&\qquad \text{(by (2.8) and (2.5))} \\
&= \langle [\xi], [-H'\delta + Q\xi] \rangle - \tfrac{1}{2}\langle [\xi], [Q\xi] \rangle \\
&\geqslant -\langle [c], [\xi] \rangle - \tfrac{1}{2}\langle [\xi], [Q\xi] \rangle \\
&\qquad \text{(by (2.6) and (2.7))} \\
&= f([\xi]).
\end{aligned}
$$

Q.E.D.

This theorem has an immediate

Corollary: For all feasible solutions to problems (P) and (D) respectively

$$\sup f([\xi]) \leq \text{Inf.}\, g([\xi], [\delta])$$

where $[\xi] \in \mathcal{T}$ and $[\delta] \in \mathcal{P}$.

Theorem 3.2. (Optimality criterion) If there exist collections $[\bar{\xi}] \in \mathcal{T}$ and $([\bar{\xi}], [\bar{\delta}]) \in \mathcal{T} \times \mathcal{P}$ feasible for problems (P) and (D) respectively such that

$$f([\bar{\xi}]) = g([\bar{\xi}], [\bar{\delta}]),$$

then $[\bar{\xi}]$ and $([\bar{\xi}], [\bar{\delta}])$ are optimal solutions to the problems (P) and (D) respectively.

Proof. By Theorem 3.1 above, for all feasible $[\xi] \in \mathcal{J}$ for the problem (P), we have

$$f([\xi]) \leq g([\bar{\xi}], [\bar{\delta}]) = f([\bar{\xi}])$$

and for all feasible $([\xi], [\delta]) \in \mathcal{J} \times \mathcal{P}$ for the problem (D), we have,

$$g([\xi], [\delta]) \geqslant f([\bar{\xi}]) = g([\bar{\xi}], [\bar{\delta}]).$$

Hence, the result follows.

Theorem 3.3. (Strong Duality) Exactly one of the following holds:

(a) If both problems (P) and (D) admit feasible solutions, then

$$\sup f([\xi]) = \operatorname{Inf} g([\xi], [\delta])$$

for $[\xi] \in \mathcal{J}$ and $[\delta] \in \mathcal{P}$.

(b) If problem (P) is feasible and (D) is not, then

$$\sup f([\xi]) = +\infty.$$

(c) If problem (D) is feasible and (P) is not, then

$$\operatorname{Inf.} g([\xi], [\delta]) = -\infty,$$

and

(d) Neither of the problems (P) and (D) is feasible.

Proof. (Part a). Let Y be any non null subset of an extended real line such that f is a continuous function from $\mathcal{J}$ to Y.

Let W be the topological product space Y x $\mathcal{P}$ and

$$S = \{(y, [\delta_1])/y \leq f([\xi]), [\delta_1] \leq [H\xi - b] \text{ for } [\xi] \geqslant [0], [\xi] \in \mathcal{J}$$

and for $[\delta_1] \geqslant [0], [\delta_1] \in \mathcal{P}\}$ be a subset of W. Then, the set S is convex (by the Lemma given below).

Let $[\bar{\xi}] \in \mathcal{J}$ be a collection for which $f([\xi])$ attains its supremum which exists by Theorem 3.1. Consider the point $w_0 = (f([\bar{\xi}]), [0]) \in W$. This point is an element of the closure of S ($= \bar{S}$) since $[0] \leq [H\bar{\xi} - b]$. Also, w_0 does not belong to the interior of $\bar{S}$, for if it does, there will exist a collection $[\xi]$ such that $f([\xi]) > f([\bar{\xi}])$ and $[0] < [H\xi - b]$, thus contradicting our optimality assumption.

By corollary to the Hahn-Banach (Bounding plane) Theorem [64], there exists a non-null continuous linear functional $w_0^* \in W$ such that

$$w_0^*(w) \leq w_0^*(w_0) \text{ for all } w \in \bar{S},$$

where $w_0^* = (y_0^*, [\delta_0^*])$ and $[\delta_0^*] \in \mathcal{D}^*$, the conjugate of $\mathcal{D}$.

Then, it follows that

$$(3.1) \quad y_0^*[f([\xi])] + \delta_0^*[\delta_1] \leq y_0^*[f([\bar{\xi}])]$$

$$\text{for all } (f([\xi]), [\delta_1]) \in \bar{S} \text{ where } [\xi] \in \mathcal{J}$$

Since $w_0 \in \bar{S}$, so do all points of the form $(f([\bar{\xi}]), [\delta_1])$ for $[\delta_1] \leq [0]$. Taking $[\xi] = [\bar{\xi}]$ in (3.1), we obtain

$$\delta_0^*[\delta_1] \leq 0 \quad \text{for all} \quad [\delta_1] \leq [0]$$

which implies that $\delta_0^*[\delta_1] \geq 0$ for all $[\delta_1] \geq [0]$ from which it follows that

$$(3.2) \qquad [\delta_0^*] \geq [0].$$

Furthermore, points of the type $(f([\xi]), [0])$ belong to the set $\bar{S}$, so that from (3.1), one gets

$$(3.3) \quad y_0^*[f([\bar{\xi}]) - f([\xi])] \geq 0.$$

Thus, from (3 is optimal for f([$\bar{\xi}$]) in (P) it follows from (3.3)

(3.10) δ_0^* 0.

, taking $y_0^* = 1$, without loss of generality, we get

Since

by the Reiss-

say [$\bar{\delta}$] $\geq$[+ $\delta_0^*[\delta_1] \leq$ f([$\bar{\xi}$])

r all [ξ] $\geq$ [0], [ξ] $\in \mathcal{J}$, and [δ_1] $\geq$ [0], [δ_1] $\in$ $\mathcal{O}$

(3.11) δ

consider the point (f([$\bar{\xi}$]) + ϵ , [0]) = $\tilde{w}$,

Since

ll positive number. The point $\tilde{w}$ does not belong $\bar{S}$.

Theorem 4.1, theorem there exists a non-null continuous linear

f([$\bar{\xi}$]) = I trictly seperates $\tilde{w}$ and $\bar{S}$. That is

replace f([

F([ξ])] + $\delta^*[\delta_1] \leq \tilde{y}^*$ [f([$\bar{\xi}$])] < $\tilde{y}^*$ [f([$\bar{\xi}$]) + ϵ]

(3.12) the conjugate space of $\mathcal{O}$.

nequality of (3.6) it follows that

for all [ξ]

From (3.12) .6) by $\tilde{y}^*$ throughout and since ϵ is arbitrarily small,

- <[b] ,[$\bar{\delta}$ m (3.5) is, again, satisfied.

[ξ] = [$\bar{\xi}$] and [δ_1] = [H ξ - b] in (3.5), it

(3.13) -b] $\leq$ 0.

Thus, from (3.8) and (3.9) one obtains that

(3.10) $\quad \delta_0^*[H\xi - b] = 0.$

Since $[\delta_0^*] \in \mathcal{P}^*$, the space of bounded linear functionals on $\mathcal{P}$, by the Reiss-representation theorem [41] there exists a unique collection, say $[\bar{\delta}] \geq [0]$, such that

(3.11) $\quad \delta_0^*[H\bar{\xi} - b] = \langle [H\bar{\xi} - b], [\bar{\delta}] \rangle = 0.$

Since $[\bar{\xi}] \in \mathcal{J}$ is optimal for the problem (P), by Theorem 4.1, below, $[\bar{\xi}]$ is also optimal for the problem (P_1) and $f([\bar{\xi}]) = F([\bar{\xi}])$. The relations (3.1) to (3.11) will also hold if we replace $f([\xi])$ by $F([\xi])$. Thus, from (3.5) we get that

$$F([\xi]) + \langle [H\xi - b], [\bar{\delta}] \rangle \leq F([\bar{\xi}]) = f([\bar{\xi}])$$

(3.12)
$$\leq -\langle [b], [\delta] \rangle + \tfrac{1}{2}\langle [\xi], [Q\xi] \rangle$$

(by Theorem 3.1)

for all $[\xi] \geq [0]$, $[\xi] \in \mathcal{J}$, and $([\xi],[\delta])$ feasible for (D).
From (3.12) and (3.11) it follows, that

$$-\langle [b], [\bar{\delta}] \rangle + \tfrac{1}{2}\langle [\bar{\xi}], [Q\bar{\xi}] \rangle + \langle [H'\bar{\delta} - Q\bar{\xi} - c], [\xi] \rangle$$

$$\leq -\langle [b], [\bar{\delta}] \rangle + \tfrac{1}{2}\langle [\bar{\xi}], [Q\bar{\xi}] \rangle$$

(3.13)
$$+ \langle [H'\bar{\delta} - Q\bar{\xi} - c], [\bar{\xi}] \rangle$$

$$-\langle [b], [\delta] \rangle + \tfrac{1}{2}\langle [\xi], [Q\xi] \rangle$$

for all $[\xi] \geq [0]$, $[\xi] \in \mathcal{J}$, $[\delta] \geq [0]$, $[\delta] \in \mathcal{P}$.

Taking $[\xi] = [\bar{\xi}]$ and $[\delta] = [\bar{\delta}]$, from the second inequality of (3.13) it follows that

$$(3.14)\quad \langle [H'\bar{\delta} - Q\bar{\xi} - c], [\bar{\xi}] \rangle \leq 0 .$$

From the first inequality of (3.13) after taking $[\xi] = [0]$, one obtains

$$(3.15)\quad \langle [H'\bar{\delta} - Q\bar{\xi} - c], [\bar{\xi}] \rangle \geq 0.$$

Thus, from (3.14) and (3.15) one gets

$$(3.16)\quad \langle [H'\bar{\delta} - Q\bar{\xi} - c], [\bar{\xi}] \rangle = 0.$$

Using (3.16) in (3.13) we see from the first inequality that

$$(3.17)\quad \langle [H'\bar{\delta} - Q\bar{\xi} - c], [\xi] \rangle \leq [0]$$

for all $[\xi] \geq [0], [\xi] \in \mathcal{J}$

from which it follows that

$$(3.18)\quad [H'\bar{\delta} - Q\bar{\xi} - c] \leq [0]$$

$$\text{and} \quad [\bar{\delta}] \geq [0] .$$

The relation (3.18) shows that $([\bar{\xi}], [\bar{\delta}])$ is feasible for the problem (D).

Now, from (3.11) and (3.16) it implies that

$$(3.19)\quad \langle [H\bar{\xi}], [\bar{\delta}] \rangle = \langle [b], [\bar{\delta}] \rangle = \langle [c], [\bar{\xi}] \rangle + \langle [\bar{\xi}], [Q\bar{\xi}] \rangle$$

which gives that

$$(3.20)\quad f([\bar{\xi}]) = -\langle [c], [\bar{\xi}] \rangle - \tfrac{1}{2} \langle [\bar{\xi}], [Q\bar{\xi}] \rangle = -\langle [b], [\bar{\delta}] \rangle + \tfrac{1}{2} \langle [\bar{\xi}], [Q\bar{\xi}] \rangle = g([\bar{\xi}], [\bar{\delta}])$$

Then, by Theorem 3.2, it implies that ($[\bar{\xi}]$, $[\bar{\delta}]$) is also optimal for the problem (D). This gives the desired result.

To prove the converse, consider the dual problem (D) as the supremum problem. Let Y be as defined earlier such that $-g([\xi], [\delta])$ is a continuous function from $\mathcal{T} \times \mathcal{O}$ to Y. Consider the topological product space $V = Y \times \mathcal{T}$, and the set $L = \{(y, [\eta]) / y \leq -g([\xi], [\delta])$
$[\eta] \leq [-H'\delta + Q\xi + c]$, for $[\delta] \geq [0], [\delta] \in \mathcal{O}$ and $[\eta] \geq [0], [\eta] \in \mathcal{T}\}$

to be the subset of V. The set L is, then, convex as established in the Lemma given below.

Proceeding on the same way with the similar arguments as above, one can easily establish the converse duality theorem.

Proof. (Part b). Let $\sup f([\xi]) < +\infty$. Then, by the proof of the part (a) we see that the existence of the optimal feasible solution to the primal, problem (P) implies and is implied by the existence of the optimal feasible to the dual problem (D). But, by the hypothesis, problem (D) has no feasible solution.

This implies that $\sup f([\xi]) = +\infty$.

Proof (part c). Similar to the proof of part (b) - above.

Lemma: The two sets S and L considered in the proof of the Theorem (3.3a) are convex.

Proof. The convexity of the set S is shown here. The proof for the set L will follow immediately.

Let $(y_1, [\delta_1])$ and $(y_2, [\delta_2])$ be any two points in S and $[\xi_1] \in \mathcal{T}$, $[\xi_2] \in \mathcal{T}$ such that $y_1 \leq f([\xi_1])$, $y_2 \leq f([\xi_2])$ and $[\delta_1] \leq [H\xi_1 - b]$ and $[\delta_2] \leq [H\xi_2 - b]$.

Now, for all $\lambda \in [0,1]$, consider the following points

$$y_\lambda = \lambda y_1 + (1-\lambda) y_2$$
$$[\delta_\lambda] = \lambda[\delta_1] + (1-\lambda)[\delta_2]$$

and

$$[\xi_\lambda] = \lambda[\xi_1] + (1-\lambda)[\xi_2]$$

Then, $[\xi_\lambda] \in \mathcal{J}$, and $[\delta_\lambda] \in \mathcal{O}$

Since, $f([\xi])$ and $[H\xi - b]$ are concave functions of $[\xi]$ we have,

$$\begin{aligned} y_\lambda = \lambda y_1 + (1-\lambda) y_2 &\leq \lambda f([\xi_1]) + (1-\lambda) f([\xi_2]) \\ &\leq f(\lambda[\xi_1] + (1-\lambda)[\xi_2]) \\ &= f([\xi_\lambda]) \end{aligned}$$

and

$$\begin{aligned} [\delta_\lambda] = \lambda[\delta_1] + (1-\lambda)[\delta_2] &\leq \lambda[H\xi_1 - b] + (1-\lambda)[H\xi_2 - b] \\ &\leq [H(\lambda\xi_1 + (1-\lambda)\xi_2) - b] \\ &= [H\xi_\lambda - b] \ . \end{aligned}$$

Thus, the point

$$\begin{aligned} (y_\lambda, [\delta_\lambda]) &= (\lambda y_1 + (1-\lambda) y_2, \lambda[\delta_1] + (1-\lambda)[\delta_2]) \\ &= \lambda(y_1, [\delta_1]) + (1-\lambda)(y_2, [\delta_2]) \end{aligned}$$

belongs to the set S, and, hence, S is convex.

<u>(The Saddle-Point) Theorem 3.4</u>. If $[\bar{\xi}]$ is a feasible solution to the problem (P), then, it is optimal for (P) if and only if there exists a collection $[\bar{\delta}] \in \mathcal{O}$, $[\bar{\delta}] \geq [0]$ such that

$$\Phi([\xi],[\bar{\delta}]) \leq \Phi([\bar{\xi}],[\bar{\delta}]) \leq \Phi([\bar{\xi}],[\delta]) \tag{3.21}$$

for all $[\xi] \in \mathcal{J}$ feasible for (P) and for all $[\delta] \in \mathcal{O}$, $[\delta] \geq [0]$.

Proof. Let $[\bar{\xi}] \in \mathcal{J}$ be an optimal feasible for the problem (P). Proceeding in the similar manner as in the proof of Theorem 3.3 (a) upto (3.11). We see that first inequality of (3.21) is satisfied.

On the other hand,

$$[H\bar{\xi} - b] \geq [0] ,$$

and $[\delta] \geq [0]$ imply that $\langle [H\bar{\xi} - b] , [\delta] \rangle \geq 0$,

and then the second inequality of (3.21) is also satisfied. Thus, there exists a collection $[\bar{\delta}] \geq [0]$, $[\bar{\delta}] \in \mathcal{Q}$ such that $([\bar{\xi}], [\bar{\delta}])$ is a saddle point for the function Φ .

To prove the converse, let $([\bar{\xi}] , [\bar{\delta}])$ be a point satisfying (3.21). Then, from the second inequality of (3.21), we get

$$(3.22) \quad f([\bar{\xi}]) + \langle [H\bar{\xi} - b] , [\bar{\delta}] \rangle \leq f([\bar{\xi}]) + \langle [H\bar{\xi} - b] , [\delta] \rangle$$

$$\text{for all } [\delta] \geq [0] \text{ and } [\delta] \in \mathcal{Q}$$

In particular, taking $[\delta] = [0]$, we get

$$(3.23) \quad \langle [H\bar{\xi} - b] , [\bar{\delta}] \rangle \leq 0$$

On the other hand

$$[H\bar{\xi} - b] \geq [0] , \text{ and } [\bar{\delta}] \geq [0] ,$$

it follows that

$$(3.24) \quad \langle [H\bar{\xi} - b] , [\bar{\delta}] \rangle \geq 0$$

Thus, by (3.23) and (3.24), it gives that

$$(3.25) \quad \langle [H\bar{\xi} - b] , [\bar{\delta}] \rangle = 0$$

Then, from the first inequality of (3.21) we obtain

$$f([\xi]) \leq f([\bar{\xi}])$$

$$\text{for all } [\xi] \geq [0], \quad [\xi] \in \mathcal{J} .$$

Thus, the theorem is proved.

Similarly, one can establish the saddle point theorem for the Lagrangian function Ψ of the dual problem.

Theorem 3.5. Let $[\bar{\xi}]$ and $([\bar{\xi}],[\bar{\delta}])$ be optimal feasible solutions for the primal and the dual problems respectively.

Then,

$$\langle [c],[\bar{\xi}]\rangle + \langle [\bar{\xi}],[Q\bar{\xi}]\rangle = \langle [b],[\bar{\delta}]\rangle$$
$$= \langle [H\bar{\xi}],[\bar{\delta}]\rangle$$

and

$$\langle [H\bar{\xi} - b],[\bar{\delta}]\rangle = 0, \quad \langle [-H'\bar{\delta} + Q\bar{\xi} + c],[\bar{\xi}]\rangle = 0.$$

Proof is immediate by the proof of Theorem 3.3 (a).

Theorem 3.6. (The converse of Theorem 3.5). Let the collections $[\bar{\xi}]\in \mathcal{T}$ and $([\bar{\xi}], [\bar{\delta}]) \in \mathcal{T} \times \mathcal{P}$ be feasible for problems (P) and (D) respectively such that

(3.26) $\quad \langle [H\bar{\xi} - b],[\bar{\delta}]\rangle = 0 \quad$ and

(3.27) $\quad \langle [H'\bar{\delta} - Q\bar{\xi} - c],[\bar{\xi}]\rangle = 0.$

Then, $[\bar{\xi}]$ and $([\bar{\xi}], [\bar{\delta}])$ are optimal for the problems (P) and (D) respectively.

Proof. Since the collections $[\bar{\xi}]\in \mathcal{T}$ and $([\bar{\xi}], [\bar{\delta}]) \in \mathcal{T} \times \mathcal{P}$ are feasible for (P) and (D) respectively satisfying (3.26) and (3.27), we have

$$\langle [H\bar{\xi}],[\bar{\delta}]\rangle = \langle [b],[\bar{\delta}]\rangle$$

a

(3.29) $\quad \langle [H\bar{\xi}],[\bar{\delta}]\rangle = \langle [c],[\bar{\xi}]\rangle + \langle [\bar{\xi}],[Q\bar{\xi}]\rangle$

By these two equalities, we get

$$f([\bar{\xi}]) = g([\bar{\xi}], [\bar{\delta}])$$

and, thus, by Theorem 3.2, $[\bar{\xi}]$ and $([\bar{\xi}], [\bar{\delta}])$ are optimal for the problems (P) and (D) respectively.

4. <u>Saddle Point Theorem and Duality.</u>

The saddle point theorem for the problems (P) and (D) respectively is established with the help of Hurwicz's Theorem V. 3.2 [64]. The treatment of the duality Theorem 3.3 and some optimality criteria are given corresponding closely to Madansky's case of linear version [77].

Now, we establish Theorem 3.4 in the following way.

Let $[\bar{\xi}] \in \mathcal{T}$ be a feasible solution to the problem (P).

Consider the following program:

$$(P_1) \quad \text{sup. } F([\xi]) = -\langle [c], [\xi] \rangle - \langle [\bar{\xi}], [Q\xi] \rangle + \tfrac{1}{2} \langle [\bar{\xi}], [Q\bar{\xi}] \rangle$$

subject to $[H\xi] \geq [b]$, $[\xi] \geq [0]$, $[\xi] \in \mathcal{T}$.

<u>Theorem 4.1.</u> Problem (P) has an optimal feasible solution $[\bar{\xi}] \in \mathcal{T}$ if and only if the problem (P_1) has an optimal feasible solution $[\bar{\xi}] \in \mathcal{T}$.

<u>Proof.</u> Notice that the constraint set for the problem (P_1) is the same as that of (P).

Now, let problem (P_1) has an optimal feasible solution $[\bar{\xi}] \in \mathcal{T}$.

Then,

$$(4.1) \quad F([\bar{\xi}]) \geq F([\xi]) \text{ for all } [\xi] \in \mathcal{T}.$$

Consider the following

$$f([\bar{\xi}]) - f([\xi]) = \langle [c], [\xi] \rangle + \tfrac{1}{2} \langle [\xi], [Q\xi] \rangle - \langle [c], [\bar{\xi}] \rangle - \tfrac{1}{2} \langle [\bar{\xi}], [Q\bar{\xi}] \rangle$$

$$\geqslant \langle [c], [\xi] \rangle - \langle [c], [\bar{\xi}] \rangle +$$
$$+ \langle [\bar{\xi}], [Q\xi] \rangle - \langle [\bar{\xi}], [Q\bar{\xi}] \rangle$$
$$= F([\bar{\xi}]) - F([\xi])$$
$$\geqslant 0 \qquad \text{(by (4.1))}$$

Thus, $[\bar{\xi}] \in \mathcal{J}$ is also optimal for the problem (P).

Now, we prove the converse by contradiction. Let $[\bar{\xi}] \in \mathcal{J}$ be an optimal feasible solution to the problem (P). Assume that there is another feasible solution $[\xi^{o}] \in \mathcal{J}$ such that

(4.2) $F([\xi^{o}]) > F([\bar{\xi}])$.

Since $[\xi^{o}]$ and $[\bar{\xi}]$ are feasible for (P), for any $0<\lambda<1$, $[\xi_{\lambda}] = \lambda [\xi^{o}] + (1-\lambda)[\bar{\xi}] \in \mathcal{J}$ and is feasible for problems (P) and (P_1). Then,

$$f([\xi_{\lambda}]) - f([\bar{\xi}]) = \langle [c], [\bar{\xi}] \rangle + \tfrac{1}{2} \langle [\bar{\xi}], [Q\bar{\xi}] \rangle$$
$$- \langle [c], [\xi_{\lambda}] \rangle - \tfrac{1}{2} \langle [\xi_{\lambda}], [Q\xi_{\lambda}] \rangle$$
$$(4.3) \qquad = \lambda [F([\xi^{o}]) - F([\bar{\xi}])]$$
$$- \frac{\lambda^2}{2} \langle [\xi^{o} - \bar{\xi}], [Q(\xi^{o} - \bar{\xi})] \rangle .$$

Since first term in the square bracket at the right hand side is independent of λ, taking λ such that

$$\lambda < \frac{F([\xi^{o}]) - F([\bar{\xi}])}{\tfrac{1}{2} \langle [\xi^{o} - \bar{\xi}], [Q(\xi^{o} - \bar{\xi})] \rangle}$$

we get from (4.3), that

$$f([\xi_{\lambda}]) - f([\bar{\xi}]) > 0$$

which contradicts the hypothesis. Thus $[\bar{\xi}] \in \mathcal{J}$ is optimal for (P_1) also.

Let $\Phi'([\xi],[\delta])$ be the Lagrangian function for the problem (P_1).

Theorem 4.2. Let the collection $[\bar{\xi}] \in \mathcal{T}$ be feasible for the problem (P_1). Then, it is optimal for (P_1) if and only if there exists a non-negative collection $[\bar{\delta}] \in \mathcal{O}$ such that

$$(4.4) \qquad \Phi'([\bar{\xi}], [\delta]) \geq \Phi'([\bar{\xi}], [\bar{\delta}]) \geq \Phi'([\xi], [\bar{\delta}])$$

for all feasible $[\xi] \in \mathcal{T}$ for (P_1) and all non-negative $[\delta] \in \mathcal{O}$.

Theorem V. 3.2 of Hurwicz [64] is used in order to prove this saddle point Theorem 4.2. To conclude this, one needs the regular convexity of the set

$$W_T^* = \{ w^* \in W^* / w^* = T^*(v^*),\ v^* \geq 0,\ v^* \in V^* \}$$

where W^* is a conjugate space of $W = R \times \mathcal{T}$ (R, the real numbers), V^* is the conjugate space of $V = \mathcal{O} \times W$ and T and T^* are linear transformations defined as follows:

$$T: W \longrightarrow V \quad \text{and } T^*: V \longrightarrow W,$$

that is, for $w = (P, [\xi]) \in W$, (P real)

$$T(P, [\xi]) = (-P[b] + [H\xi], (P, [\xi]))$$
$$= ([\delta], P, [\xi]) \in V,$$

and

$$T^*([\delta], P, [\xi]) = (P, [\xi]) \in W.$$

Now, the space W is a Hilbert space with respect to the inner product

$$\langle (P_1, [\xi_1]), (P_2, [\xi_2]) \rangle = P_1 P_2 + \langle [\xi_1], [\xi_2] \rangle .$$

The space V is a Hilbert space with respect to the inner product

$$\langle ([\delta_1], P_1, [\xi_1]), ([\delta_2], P_2, [\xi_2]) \rangle$$

$$= \langle [\delta_1], [\delta_2] \rangle + P_1 P_2 + \langle [\xi_1], [\xi_2] \rangle .$$

Thus, $W = W^*$ and $V = V^*$.

Let $v = ([\delta], P, [\xi])$

and $v_1 = T(w_1) = (-P_1 [b] + [H\xi_1], (P_1, [\xi_1]))$.

Then, $w_1 = (P_1, [\xi_1])$, and

$$\langle v, v_1 \rangle = \langle v, Tw_1 \rangle = \langle T^* v, w_1 \rangle .$$

But,

$$\langle v, Tw_1 \rangle = -P_1 \langle [\delta], [b] \rangle + \langle [\delta], [H\xi_1] \rangle + P P_1 + \langle [\xi], [\xi_1] \rangle$$

$$= (P - \langle [\delta], [b] \rangle) P_1 + \langle [H'\delta + \xi], [\xi_1] \rangle$$

$$= \langle T^* v, w_1 \rangle ,$$

i.e.

$$T^*([\delta], P, [\xi]) = (P - \langle [\delta], [b] \rangle, [H'\delta + \xi])$$

belongs to W.

Now, to see That W_{T^*} is regularly convex, we have to show that for a given $w_o^* \in W^*$ and $w_o^* \notin W_T^*$, there exists a $w_o \in W_{T^*}$ such that

$$\sup_{w^* \in W^*} w^*(w_o) < w_o^*(w_o)$$

or equivalently to showing that W_{T^*} is convex and weak* closed.

To show that W_{T^*} is convex, let $w_1^* \in W_{T^*}$ and $w_2^* \in W_{T^*}$ such that $w_1^* = T^*(v_1^*)$, $w_2^* = T^*(v_2^*)$, $v_1^* \geqslant 0$, $v_2^* \geqslant 0$, $v_1^*, v_2^* \in V^*$. Since V^* is convex, we have for any $\lambda \in [0,1]$,

$$v_\lambda^* = \lambda v_1^* + (1-\lambda) v_2^* \geqslant 0, \text{ and } v_\lambda^* \in V^*$$

Since T^* is a linear transformation we have

$$T^*(v^*_\lambda) = T^*(\lambda v^*_1 + (1-\lambda)v^*_2)$$

$$= T^*(\lambda v^*_1) + T^*((1-\lambda) v^*_2)$$

$$= \lambda T^*(v^*_1) + (1-\lambda) T^*(v^*_2)$$

$$= \lambda w^*_1 + (1-\lambda) w^*_2$$

$$= w^*_\lambda \quad .$$

Thus, $w^*_\lambda \in W_{T^*}$, i.e., W_{T^*} is convex.

To know that W_{T^*} is weak* closed, consider a sequence $\{w_n\} \subset W_{T^*}$ such that $w_n \to w$ in the weak* topology We must show that $w \in W_{T^*}$, that is, for $w_n = (P_n - \langle [b], [\delta_n] \rangle , [H'\delta_n + \xi_n])$,

$$\lim_{n\to\infty} (\nu P_n - \nu \langle [b], [\delta_n]\rangle + \langle [\eta], [H'\delta_n + \xi_n]\rangle)$$

$$= \lim_{n\to\infty} \langle (\nu, [\eta]), w_n \rangle \quad \text{for all } (\nu, [\eta]) \in W,$$

$$= (\nu P - \nu \langle [b], [\delta]\rangle + \langle [\eta], [H'\delta + \xi]\rangle)$$

$$= \langle (\nu, [\eta]), w \rangle$$

where $P_n \to P$ numerically and $[\delta_n] \to [\delta]$ and $[\xi_n] \to [\xi]$ in the appropriate weak* topologies. But

$$|\langle (\nu, [\eta]), w\rangle| \le |\nu P| + |\nu\langle [b], [\delta]\rangle| + |\langle [\eta], [H'\delta + \xi]\rangle|$$

Since μ is a probability measure, and using the Schwarz inequality, $\langle [b], [\delta]\rangle$ is bounded by the constant function $\sqrt{\langle [b], [b]\rangle \langle [\delta] [\delta]\rangle}$ and similarly $\langle [\eta], [H'\delta + \xi]\rangle$, so that using the dominated convergence

theorem, T^* is continuous and the result follows,

i.e. $w = (P - \langle [b], [\delta] \rangle , [H'\delta + \xi]) \in W_{T^*}$

Thus, the Theorem 4.2 is proved.

Theorem 4.3. The point $([\xi],[\delta]) \in \mathcal{T}\times\mathcal{O}$ is a saddle point for the function $\Phi' \iff$ the point $([\bar{\xi}], [\bar{\delta}])$ is a saddle point for the function Φ.

Proof. Let $([\bar{\xi}], [\bar{\delta}])$ be the saddle point for the function Φ'.

Then, since

$$(4.5) \quad \Phi'([\bar{\xi}], [\delta]) = -\langle [c], [\bar{\xi}] \rangle - \tfrac{1}{2}\langle [\bar{\xi}], [Q\bar{\xi}] \rangle + \langle [H\bar{\xi} - b], [\delta] \rangle$$
$$= \Phi([\bar{\xi}], [\delta]) \text{ for all } [\delta] \geq [0] \text{ and } [\delta] \in \mathcal{O}$$

$$(4.6) \quad \Phi'([\bar{\xi}], [\bar{\delta}]) = \Phi([\bar{\xi}], [\bar{\delta}])$$

and

$$(4.7) \quad \Phi'([\xi], [\bar{\delta}]) = -\langle [c], [\xi] \rangle - \langle [\xi], [Q\bar{\xi}] \rangle + \tfrac{1}{2}\langle [\bar{\xi}], [Q\bar{\xi}] \rangle + \langle [H\xi - b], [\bar{\delta}] \rangle$$
$$\geq -\langle [c], [\xi] \rangle - \tfrac{1}{2}\langle [\xi], [Q\xi] \rangle + \langle [H\xi - b], [\bar{\delta}] \rangle$$

(by the property of Q)

$$= \Phi([\xi], [\bar{\delta}]).$$

Thus, by (4.5), (4.6) and (4.7), we get

$$(4.8) \quad \Phi([\xi], [\bar{\delta}]) \leq \Phi([\bar{\xi}], [\bar{\delta}]) \leq \Phi([\bar{\xi}], [\delta])$$

for all $[\xi] \in \mathcal{T}$, feasible for (P) and $[\delta] \geq [0]$, $[\delta] \in \mathcal{O}$.

Conversely, let (4.8) be satisfied. Then, by the converse part of the Theorem 3.4, $[\bar{\xi}] \in \mathcal{T}$ is an optimal solution for the problem (P) and by Theorem 4.1, this is also optimal for the problem (P_1). Thus, by Theorem 4.2, there exists a $[\bar{\delta}] \geq [0]$, $[\bar{\delta}] \in \mathcal{P}$ such that $([\bar{\xi}], [\bar{\delta}])$ is a saddle point for the Lagrangian function Φ'. Hence, the theorem is proved.

Notice that Theorem 3.4 is established as a consequence of Theorems 4.1, 4.2 and the first part of the Theorem 4.3. The same mode of proof establishes the saddle point theorem for the Lagrangian function Ψ of the dual problem (D).

Consider the following:

$$[z] = ([\xi], [\delta]),\ [d] = ([0], [b]),\ M = [Q, -H] \text{ and } N = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}$$

The problem (D) reduces to the following form:

$$\text{(D')} \quad \sup.\ \langle [d], [z] \rangle - \tfrac{1}{2} \langle [z], [Nz] \rangle$$

$$\text{subject to} \quad [Mz + c] \geq [0],\ [z] \in \mathcal{T} \times \mathcal{P}$$

$$[\delta] \geq [0],\ [\delta] \in \mathcal{P}$$

The Lagrangian function for this problem can be written as follows:

$$(4.10) \quad \Psi'([z], [\xi]) = \langle [d], [z] \rangle - \tfrac{1}{2} \langle [z], [Nz] \rangle + \langle [Mz + c], [\xi] \rangle$$

__Theorem 4.4.__ If $([\bar{\xi}], [\bar{\delta}]) \in \mathcal{T} \times \mathcal{P}$ is a saddle point for the function Φ, then $([\bar{z}], [\bar{\xi}])$ is a saddle point for the function Ψ' where $[\bar{z}] = ([\bar{\xi}], [\bar{\delta}])$ and conversely.

__Proof.__ Let (4.8) be satisfied for all $[\xi] \in \mathcal{T}$ feasible for (P) and all $[\delta] \in \mathcal{P}$. Then,

$$\Phi([\bar{\xi}], [\delta]) = -\langle[c], [\bar{\xi}]\rangle - \tfrac{1}{2}\langle[\bar{\xi}], [Q\bar{\xi}]\rangle$$
$$+ \langle[H\bar{\xi} - b], [\delta]\rangle$$
$$\leq -\langle[c], [\bar{\xi}]\rangle + \tfrac{1}{2}\langle[\xi], [Q\xi]\rangle$$

(4.11) $$-\langle[\xi], [Q\bar{\xi}]\rangle + \langle[H\bar{\xi} - b], [\delta]\rangle$$

(by the property of Q)

$$= -\langle[b], [\delta]\rangle + \tfrac{1}{2}\langle[\xi], [Q\xi]\rangle$$
$$+ \langle[H'\delta - Q\xi - c], [\bar{\xi}]\rangle$$
$$= -\Psi'([s], [\bar{\xi}])$$

and

(4.12) $$\Phi([\bar{\xi}], [\bar{\delta}]) = -\Psi'([\bar{s}], [\bar{\xi}]).$$

Furthermore,

$$\Phi'([\xi], [\bar{\delta}]) = -\langle[c], [\xi]\rangle + \tfrac{1}{2}\langle[\bar{\xi}], [Q\bar{\xi}]\rangle$$
$$- \langle[\xi], [Q\bar{\xi}]\rangle + \langle[H\xi - b], [\bar{\delta}]\rangle$$
$$= -\langle[b], [\bar{\delta}]\rangle + \tfrac{1}{2}\langle[\bar{\xi}], [Q\bar{\xi}]\rangle + \langle[H'\bar{\delta} - Q\bar{\xi} - c], [\xi]\rangle$$
$$= -\Psi'([\bar{s}], [\xi]).$$

Then, by (4.11), (4.12), (4 13) and Theorem 4.3, we obtain

(4.14) $$\Psi'([\bar{s}], [\xi]) \geq \Psi'([\bar{s}], [\bar{\xi}]) \geq \Psi'([s], [\bar{\xi}]),$$

for all $[\xi] \geq [0]$, $[\xi] \in \mathcal{T}$, and all $[\delta] \geq [0]$, $[\delta] \in \mathcal{P}$, and conversely.

As a consequence of these theorems and Theorem IV 3 of [64] the duality theorem can be established. In the present setting, Theorem IV. 3 of [64] states the following:

If $\exists\, [\xi_1] \in \mathcal{T}$, $[\xi_1] \geq [0]$, $[H\xi_1] \geq [b]$

and $[\xi_2] \in \mathcal{T}$, such that

$$\langle[\xi_2], [\xi]\rangle \geq \beta, \ \forall\, [\xi] \geq [0],$$

$$[H\xi] \geq [b], \quad [\xi] \in \mathcal{T},$$

then, $\exists\ [\delta] \in \mathcal{P}$, $[\delta] \geq [0]$, such that

$$\langle [\delta], [H\xi - b] \rangle \leq \langle [\xi_2]\ [\xi] \rangle - \beta$$
$$\forall [\xi] \geq [0],\ [\xi] \in \mathcal{T}$$

Further, for $[\xi] \in \mathcal{T}$, such that

$$[H\xi] \geq [b], \quad [\xi] \geq [0] \text{ and } \langle [\xi_2]\ [\xi] \rangle = \beta$$

$$\langle [\delta], [H\xi - b] \rangle = 0$$

<u>Theorem (4.5)</u> (Complementary Slackness) Let $[\bar{\xi}] \in \mathcal{T}$ and $([\bar{\xi}], [\bar{\delta}]) \in \mathcal{T} \times \mathcal{P}$ be optimal feasible for the problems (P) and (D) respectively

Then

$$\langle [c], [\bar{\xi}] \rangle + \langle [\bar{\xi}], [Q\bar{\xi}] \rangle = \langle [b], [\bar{\delta}] \rangle$$
$$= \langle [H\bar{\xi}], [\bar{\delta}] \rangle$$

and

$$\langle [H\bar{\xi} - b], [\bar{\delta}] \rangle = \langle [H'\bar{\delta} - Q\bar{\xi} - c], [\bar{\xi}] \rangle = 0.$$

Notice that this is Theorem 3.5

<u>Proof</u>. Let $[\xi_1] = [\bar{\xi}]$, $[\xi_2] = [c + \frac{1}{2}Q\bar{\xi}]$ and

$$\beta = \langle [c + \tfrac{1}{2}Q\bar{\xi}], [\bar{\xi}] \rangle.$$

Since $[\bar{\xi}]$ is optimal for the problem (P), by Theorem 4.1, it is also optimal for the problem (P_1) By Theorem 4.2, we see that

$$\beta \leq \langle [c], [\bar{\xi}] \rangle + \langle [\bar{\xi}], [Q\bar{\xi}] \rangle - \tfrac{1}{2} \langle [\bar{\xi}], [Q\bar{\xi}] \rangle.$$

Then, by Theorem IV 3 of [64] $\exists\ [\delta_1] \geq [0]$, $[\delta_1] \in \mathcal{P}$

such that

(4.15) $$\langle [\delta_1], [H\bar{\xi} - b] \rangle = 0.$$

The relation (4.15) together with the following

$$\Phi\left([\bar{\xi}],[\delta_1]\right) \geq \Phi\left([\bar{\xi}],[\bar{\delta}]\right) \text{ (from (4.8))}$$

implies that

(4 16) $\quad \langle [H\bar{\xi} - b], [\bar{\delta}] \rangle \leq 0.$

But, $\quad [H\bar{\xi} - b] \geq [0]$ and $[\bar{\delta}] \geq [0]$ imply that

(4 17) $\quad \langle [H\bar{\xi} - b], [\bar{\delta}] \rangle \geq 0$

By (4.16) and (4 17), we get

$$\langle [H\bar{\xi} - b], [\bar{\delta}] \rangle = 0$$

and $\quad \langle [H\bar{\xi}], [\bar{\delta}] \rangle = \langle [b], [\bar{\delta}] \rangle .$

By the same mode of arguments and using saddle point theorem for the Lagrangian functions Ψ for the dual problem (D), it follows that

$$\langle [H\bar{\xi}], [\bar{\delta}] \rangle = \langle [c], [\bar{\xi}] \rangle + \langle [\bar{\xi}], [Q\bar{\xi}] \rangle$$

and

$$\langle [H'\bar{\delta} - Q\bar{\xi} - c], [\bar{\xi}] \rangle = 0.$$

Hence, the theorem is proved.

The converse of this theorem is stated and proved in the section 3.

<u>Proof of Theorem 3.3(a)</u> Let $[\bar{\xi}] \in \mathcal{J}$ be an optimal feasible solution to the problem (P). Then, by the consequences of the above theorems of this section it follows that $([\bar{\xi}], [\bar{\delta}])$ is an optimal feasible solution for the problem (D), and the extreme values of the two problems (P) and (D) are equal. The similar argument yields the proof of the converse of this theorem.

5. THE SECOND-STAGE PROGRAM.

We now consider the second-stage program

(5.1) Minimize $q'y + \frac{1}{2} y'Dy$

subject to $By \geq b-Ax$, for almost all b

$y \geq 0$

and its dual problem

(5.2) Maximize $(b-Ax)'\pi - \frac{1}{2} y'Dy$

subject to $B'\pi - Dy \leq q$

$\pi \geq 0$

Let, for given b and x, $\bar{y}(b,x)$ and $(\bar{y}(b,x), \bar{\pi}(b,x))$ be the optimal feasible solutions to the problems (5.1) and (5.2) respectively. Then, the following results are the consequences of the results obtained in the previous sections.

Theorem 5.1. Suppose that $x = \bar{x}$. Let $((\bar{x},\bar{y}(b,\bar{x})), \bar{\pi}(b,\bar{x}))$ be an optimal feasible dual vector for the second-stage program such that

(5.3) $(p + C\bar{x} - \bar{\pi}'(b,\bar{x}) A)' \bar{x} = 0$

and

(5.4) $(p + C\bar{x} - \bar{\pi}'(b,\bar{x}) A) \geq 0$ for almost all b

Then, $[\bar{\xi}(b)] = [\bar{x}, \bar{y}(b,\bar{x})]$

and $([\bar{\xi}(b)], [\bar{\delta}(b)]) = ([\bar{x}, \bar{y}(b,\bar{x})], [\bar{\pi}(b,\bar{x})])$

are optimal feasible for the original programs.

Proof. Since, the vectors in $((\bar{x},\bar{y}(b,\bar{x}), \bar{\pi}(b,\bar{x}))$ are optimal feasible for the dual program (5.2), they satisfy

(5.5) $(q + D\bar{y}(b,\bar{x}) - \bar{\pi}'(b,\bar{x})B) \geq 0$, and

(5.6) $(q + D\bar{y}(b,\bar{x}) - \bar{\pi}'(b,\bar{x})B)' \bar{y}(b,\bar{x}) = 0$ for almost all b

Then, by Theorem 3.6, it follows from (5.3) to (5.6), that

$$\lfloor \bar{\xi}(b)\rfloor = [\bar{x}, \bar{y}(b,\bar{x})]$$

and $\quad (\lfloor \bar{\xi}(b)\rfloor, \lfloor \bar{\delta}(b)\rfloor) = ([\bar{x},\bar{y}(b,\bar{x})], \lfloor \bar{\pi}(b,\bar{x})\rfloor)$

are optimal for the original programs.

The proof is, thus, complete

<u>Theorem 5.2</u>. Let $[\bar{\xi}(b)] = \lfloor \bar{x}, \bar{y}(b,\bar{x})\rfloor$ be an optimal feasible for the primal problem (P) Then, there exists a collection $\lfloor \bar{\delta}(b)\rfloor = \lfloor \bar{\pi}(b,\bar{x})\rfloor$, $\lfloor \bar{\pi}(b,\bar{x})\rfloor \geqslant \lfloor 0 \rfloor$, such that the collection $(\lfloor \bar{x},\bar{y}(b,\bar{x})\rfloor, \lfloor \bar{\pi}(b,\bar{x})\rfloor)$ satisfies the following:

$$(5.7) \qquad (p + C\bar{x} - \bar{\pi}'(b,\bar{x})\, A) \geqslant 0,$$

and

$$(5.8) \qquad (p + C\bar{x} - \bar{\pi}(b,\bar{x})\, A)'\, \bar{x} = 0 \quad \text{for almost all } b.$$

<u>Proof</u>. Since $\lfloor \bar{\xi}(b)\rfloor = [\bar{x},\bar{y}(b,\bar{x})]$ is optimal feasible for the problem (P), by the duality Theorem 3.3(a), there exists a $\lfloor \bar{\delta}(b)\rfloor = [\bar{\pi}(b,\bar{x})] \geqslant [0]$, such that $([\bar{\xi}(b)], [\bar{\delta}(b)]) = ([\bar{x},\bar{y}(b,\bar{x})], [\bar{\pi}(b,\bar{x})])$ is optimal feasible for the problem (D) and satisfies the results of Theorem 3.5 or Theorem 4.6. That is, to say, $([\bar{x},\bar{y}(b,\bar{x})], [\bar{\pi}(b,\bar{x})])$ satisfies (5.7), (5.8) and the following

$$(5.9) \qquad q + D\bar{y}(b,\bar{x}) - \bar{\pi}'(b,\bar{x})\, B \geqslant 0$$

and

$$(5.10) \qquad (q + D\bar{y}(b,\bar{x}) - \bar{\pi}'(b,\bar{x})\, B)'\, \bar{y}(b,\bar{x}) = 0$$

for almost all b.

Thus, the Theorem is proved.

CHAPTER - IV

A CLASS OF STOCHASTIC PROGRAMMING PROBLEMS*

1. INTRODUCTION

A class of stochastic programming problems with various types of models, is considered in this Chapter. The following types of programming models are studied:

(a) Stochastic linear programming problems with linear losses

(b) Stochastic linear programming problems with quadratic losses

(c) Quadratic programming problems under risk with linear losses

where all the models (a), (b) and (c) are subject to the linear deterministic as well as probabilistic constraints.

and

(d) Stochastic linear programming problems with linear losses

---

* Contents of this chapter are based on the paper [118] which is accepted for the publication.

subject to a probabilistic quadratic constraint and linear deterministic as well as probabilistic constraints.

All these problems consist of two stage formulations. A first-stage problem is that in which optimization is performed over a programming model without having the prior knowledge of the random outcomes or without making observations over random variables. In doing so, there occur some inaccuracies which can be noticed after the random variables have been observed. Even, if the distributions of random variables or their prior information is available, discrepancies may occur in the solution of the program. These inaccuracies or discrepancies are, then, compensated in another program called the second stage program. In a similar fashion, these problems can be generated to n-stage formulations.

It is assumed that the distributions of the random variables are known. It is, also, assumed that random variables possess finite means and variances. Some necessary and/or sufficient conditions for the existence of the finite optimum of the programs i.e. for programs to be proper, are obtained.

The importance of the above problems is that they all exhibit the similar type of decision equivalent deterministic programs. These deterministic programs come out to be convex programs with non-differentiable objective functions. Solutions of such problems are difficult to obtain by the available techniques of mathematical programming. But duality theorem can help us a lot to overcome this difficulty. The dual of these certainty equivalent programs turn out to be concave quadratic differentiable programs. The available methods for solving quadratic programming problems [14a, 73a] can be applied to these dual programs to find the optimum solutions which, in turn, are the optimum solutions of primal problems.

The corresponding term $f's + \|Ds\|$ in the objective is the loss function due to this uncertainty. $\|\ \|$ denotes the usual Euclidean norm. Prime denotes the transpose.

Let $\eta_i(x) = T_i x - a_i$, $i=1,2,\text{------}m_1$.

Now, if T_i and a_i are normally distributed, $\eta_i(x)$ being a linear combination of normal variates, it is normally distributed with mean

$$\overline{\eta_i(x)} = E(\eta_i(x))$$

and variance

$$V_{\eta_i(x)} = E\left[\eta_i(x) - \overline{\eta_i(x)}\right]^2 \quad , i=1,2,\text{------}m_1 .$$

We, then, can find out the values of t_i's such that

$$P\left[\eta_i(x) + W_i y \geqslant \overline{\eta_i(x)} - t_i\, \sigma_{\eta_i(x)} + W_i y\right]$$

$$= P\left[\frac{\eta_i(x) - \overline{\eta_i(x)}}{\sigma_{\eta_i(x)}} \geqslant -t_i\right] = \alpha_i \text{ , } i=1,2,\text{------}m_1$$

where $\sigma_{\eta_i(x)}$ stands for the standard deviation of $\eta_i(x)$.

Then, for (1.2) to be satisfied i.e., for the constraints

$$P\left[\eta_i(x) + W_i y \geqslant 0\right] \geqslant \alpha_i \text{ , } \quad i=1,2,\text{------}m_1$$

to hold, we can write an equivalent relation

$$\overline{\eta_i(x)} - t_i\, \sigma_{\eta_i(x)} + W_i y \geqslant 0$$

or

$$\overline{T_i} x - t_i\, (x'B^i x)^{\frac{1}{2}} + W_i y \geqslant a_i^* = \overline{a_i} - t_i \sigma_{a_i} \quad , \qquad i=1,2,\text{------}m_1$$

where $\overline{T_i x}$ is the expected value of $T_i x$ and $(xB^i x)^{\frac{1}{2}}$ is a standard deviation of $T_i x$, B^i for each i, $i=1,2,\ldots,m_1$ is a non-negative definite symmetric matrix, $\overline{a}_i = E(a_i)$ and σa_i = variance of a_i.

If T_i and a_i have known finite means and variances, by the Chebyshev inequality we have

$$P\left[\frac{\eta_i(x) - \overline{\eta_i(x)}}{-\sigma\bar{\eta}_i(x)} \geq t_i\right] \leq \frac{1}{t_i^2} \quad , i = 1,2,\ldots,m_1$$

That is

$$(2.4) \quad P\left[\eta_i(x) + W_i y \geq \overline{\eta_i(x)} - t_i\, \sigma_{\eta_i(x)} + W_i y\right] \geq \frac{t_i^2 - 1}{t_i^2}$$

$$i=1,2,\ldots,m_1,$$

where $t_i > 0$ and $\sigma_{\bar{\eta}_i(x)} \geq 0$.

Taking $t_i^2 = \frac{1}{1-\alpha_i}$, $i=1,2,\ldots,m_1$, (2.4) reduces to

$$P\left[\eta_i(x) + W_i y \geq \overline{\eta_i(x)} - t_i\, \sigma_{\eta_i(x)} + W_i y\right] \geq \alpha_i$$

$$i=1,2,\ldots,m_1$$

and this condition is satisfied if

$$(2.5) \quad \overline{T_i x} - t_i(x' B^i x)^{\frac{1}{2}} + W_i y \geq a_i^* = \overline{a}_i - t_i\, \sigma a_i$$

$$i=1,2,\ldots,m_1$$

i.e.,

$$(2.6) \quad \overline{T_i x} - (x' V^i x)^{\frac{1}{2}} + W_i y \geq a_i^* \;, \qquad i=1,2,\ldots,m_1$$

where $V^i = t_i^2 B^i$, $i=1,2,\text{------}m_1$.

Since the constraints (1.3) are satisfied with probability one, we can write them as

(2.7) $Mx + Nz \geqslant q$, almost all q and M.

Programming model (2.1) to (2.3) is a two-stage program. The second stage program can be split up into two programs for given x, M and q, namely

(2.8) $\underset{y \geqslant 0}{\text{Minimise}} \quad d'y$

subject to $\quad W_i y \geqslant a_i^* - \overline{T_i}x + (x'V^ix)^{\frac{1}{2}} \qquad i=1,2,\text{------}m_1$

and

(2.9) $\underset{z \geqslant 0}{\text{Minimise}} \quad f'z + \|Dz\|$

subject to $\quad Nz \geqslant q-Mx$.

Since the program (2.8) is linear in y, we can express its value by

$$L(x) = \underset{y \geqslant 0}{\text{Min}} \left\{ d'y / W_i y \geqslant a_i^* - \overline{T}_i x + (x'V^ix)^{\frac{1}{2}},\ i=1,2,\text{------}m_1 \right\}$$

(2.10) $$= \sum_{i=1}^{m_1} p_i(a_i^* - \overline{T}_i x + (x'V^ix)^{\frac{1}{2}}), \text{ say.}$$

The program (2.9) can be expressed as follows:

$$\nu(q-Mx) = \underset{z \geqslant 0}{\text{Min}} \left\{ f'z + \|Dz\| \,/\, Nz \geqslant q-Mx \right\}$$

(2.11) $$= g'(q-Mx) + \|G\,(q-Mx)\|, \text{ say,}$$

where g and G are $m_2 \times 1$ and $l \times m_2$ matrices respectively. The programming

model (2.1) to (2.3) can, then, be expressed as follow

$$(2.12)\quad \underset{x\geq 0}{\text{Min}}\ E\Big[c'x + \sum_{i=1}^{m_1} p_i(a_i^* - \overline{T}_i x + (x'V^i x)^{\frac{1}{2}}) + g(q-Mx)$$

$$+ \| G\ (q-Mx) \| \Big]$$

subject to $Ax \geq b$.

Program (2.12) is said to be proper if it obtains a finite optimum. In other way, if the expected value of c and means and covariances of a,q,M and T_i i=1,2,------m_1 are finite, then the program (2.1) to (2.3) is said to be proper if the program

$$(2.13)\quad \underset{x\geq 0}{\text{Min}}\ E\Big[(c- \sum_{i=1}^{m_1} p_i\overline{T}_i - g'M)'x + \sum_{i=1}^{m_1} (x'Q^i x)^{\frac{1}{2}}$$

$$+ \|G(g-Mx)\| + \sum_{i=1}^{m_1} p_i a_i^* + g'q\Big]$$

subject to $Ax \geq b$

where $Q^i = p_i^2\ V^i$, is proper. Since $\sum_{i=1}^{m_1} p_i a_i^*$ is finite and $E\ (g'q)$ is finite, the program (2.13) is proper if the function

$$(2.14)\quad F(x) = E\Big((c- \sum_{i=1}^{m_1} p_i\overline{T}_i - g'M)'x + \sum_{i=1}^{m_1} (x'Q^i x)^{\frac{1}{2}} + \|G(q-Mx)\|\Big)$$

is bounded below. In particular, if $Q^i x = 0$ and $G = 0$, then (2.14) will be proper if $(Ec - \sum_{i=1}^{m_1} p_i\overline{T}_i - g'EM)'x \geq 0$ in the trivial case or otherwise x should be further constrained. Though the set $\{x/Ax \geq b,\ x \geq 0\}$ is a convex polyhedral, yet the function F(x) may obtain unbounded solution on this set. But we need the finite value of

the program. If the set over which the convex function is defined, is bounded, then the function is bounded below [54]. We shall give, below, a sufficient condition for the function of the above program over the whole space R^n.

Let $Ec - \sum_{i=1}^{m_1} p_i \bar{T}_i - g'(EM) = h$. The problem (2.13) then, becomes

$$(2.15) \quad \underset{x \geq 0}{\text{Min}}\ F(x) = h'x + \sum_{i=1}^{m_1} (x'Q^i x)^{\frac{1}{2}} + E\ \|G(q-Mx)\|$$

subject to $Ax \geq b$

where constant terms are dropped. The program (2.15) is called the deterministic equivalent program for the problem (2.1) to (2.5).

Lemma 1. The function $F(x)$ in the program (2.15) is convex over any convex subset of R^n.

Proof. Let $S \subset R^n$ be a convex set. Let $x_1, x_2 \in S$.

Then, for all $\lambda \in [0,1]$

$$x_\lambda = \lambda x_1 + (1-\lambda)x_2 \in S.$$

Now,

$$F(x_\lambda) = h'x_\lambda + \sum_{i=1}^{m_1} (x'_\lambda Q^i x_\lambda)^{\frac{1}{2}} + E\,\|G(q-Mx_\lambda)\|$$

$$= h'(\lambda x_1 + (1-\lambda)x_2) + \sum_{i=1}^{m_1} \Big[\lambda^2 (x'Q^i x_1) + 2\lambda(1-\lambda)x_1 Q^i x_2$$

$$+ (1-\lambda)^2 x_2' Q^i x_2\Big]^{\frac{1}{2}} + E\,\|\lambda G(q-Mx_1) + (1-\lambda)G(q-Mx_2)\|$$

By the property of norm, and since, the matrix Q^i, $i=1,2,\text{---}m_1$ is a non-negative definite symmetric matrix, using a well known result

$$(2.16) \qquad x_1' Q^i x_2 \le (x_1' Q^i x_1)^{\frac{1}{2}} \, (x_2' Q^i x_2)^{\frac{1}{2}}$$

proof for which can be seen in [129a] , we see that

$$F(x_\lambda) \le \lambda h' x_1 + (1-\lambda) h' x_2 + \sum_{i=1}^{m_1} \left[\lambda (x_1' Q^i x_1)^{\frac{1}{2}} + (1-\lambda)(x_2' Q^i x_2)^{\frac{1}{2}} \right]$$

$$+ \lambda E \| G(q - Mx_1) \| + (1-\lambda) E \| G(q - Mx_2) \|$$

$$= \lambda F(x_1) + (1-\lambda) F(x_2).$$

Thus, the lemma follows.

Theorem 2.1. Let the expectation of c and means and covariances of q and M be finite. Then, either of the following two conditions is sufficient for the program (2.1)-(2.3) to be proper:

(i) $\exists$ vectors $u \in R^\ell$ and $v^i \in R^n$ $(i=1,2,\ldots,m_1)$ with $\|u\| \le 1$

and $v^i \geq 0$, $v^{i'} Q^i v^i \le 1$, $(i=1,2,\ldots,m_1)$, such that

$$h + \sum_{i=1}^{m_1} Q^i v^i \geq E(M' G' u)$$

or

(ii) The n x n matrix $E(M'G'GM) - \sum_{i=1}^{m_1} Q^i - h\,h'$

is a non-negative definite.

Proof. We shall show that under (i) or (ii) the function $F(x)$ is bounded below.

Let condition (i) be satisfied. By premultiplying this relation by $x \geq 0$, we obtain

$$h'x + \sum_{i=1}^{m_1} x' Q^i v^i \geq E(x' M' G' u)$$

Using (2.16) and $v^{i'}Q^i v^i \leq 1$, we see that

$$h'x + \sum_{i=1}^{m_1} (x'Q^i x)^{\frac{1}{2}} \geq E(x'M'G'u)$$

that is,

$$(2.17)\quad h'x + \sum_{i=1}^{m_1} (x'Q^i x)^{\frac{1}{2}} + E\,\|MGx\| \geq E\,\|MGx\| + E(x'M'G'u)$$

Under the given condition, we know that

$$(2.18)\quad E\,|x'M'G'u| \leq E\,\|MGx\|\ \|u\| \leq E\,\|MGx\|$$

which implies that, whatever be the sign of $E(x'M'G'u)$,

$$(2.19)\quad E\,\|MGx\| + E(x'M'G'u) \geq 0.$$

Thus, from (2.17) and (2.19) one obtains

$$(2.20)\quad h'x + \sum_{i=1}^{m_1} (x'Q'x)^{\frac{1}{2}} + E\,\|GMx\| \geq 0$$

Now,

$$F(x) = h'x + \sum_{i=1}^{m_1} (x'Q^i x)^{\frac{1}{2}} + E\,\|G\,(q-Mx)\|$$

$$(2.21)\quad \geq h'x + \sum_{i=1}^{m_1} (x'Q^i x)^{\frac{1}{2}} + E\,\|MGx\| - E\,\|Gq\|$$

$$\geq -E\,\|Gq\| \qquad \text{(by (2.20))}.$$

Notice that covariance matrix of q exists, its trace, hence $E\,\|q\|^2$, and thus $E\,\|q\|$, also exists. Now, if $\|G\|$ is the Euclidean vector norm for G, we have

$$E\,\|Gq\| \leq \|G\|\ E\,\|q\| < \infty$$

This implies that $F(x)$ is bounded below. Thus, the result follows.

Now, suppose that the condition (ii) is satisfied. Then we have

$$0 \leq E(x'M'G'GMx) - \sum_{i=1}^{m_1} (x'Q^i x) - x'hh'x$$

$$\leq E \| MGx \|^2 - x'hh'x$$

$$\leq E \| MGx \|^2 + x'Q^i x - x'hhx$$

$$(\text{since } x'Q^i x \geq 0)$$

$$\leq E \| MGx \|^2 + 2E \| MGx \| (x'Q^i x)^{\frac{1}{2}} + (x'Q^i x) - x'hh'x$$

$$(\text{Since } E \| MG\, x \| \geq 0, \text{ and } (x'Q^i x)^{\frac{1}{2}} \geq 0)$$

$$= \left[E \| MGx \| + (x'Q^i x)^{\frac{1}{2}} + h'x\right]\left[E \| MGx \| + (x'Q^i x)^{\frac{1}{2}} - h'x\right]$$

This implies that either (a) both factors are non-positive or (b) both factors are non-negative. If (a) is satisfied, then the second factor will give $h'x \geq E \| MGx \| + (x'Q^i x)^{\frac{1}{2}}$ which shows that $h'x \geq 0$. This gives that $h'x + E \| MGx \| + (x'Q^i x)^{\frac{1}{2}} \geq 0$. But, this factor is non-positive, by the condition (a). This implies that $h'x + E \| MGx \| + (x'Q^i x)^{\frac{1}{2}} = 0$ which is satisfied only for the trivial case i.e. for $x = 0$. But $x \neq 0$, and $x \geq 0$, in which case $h'x + E \| MGx \| + (x'Q^i x)^{\frac{1}{2}}$ is non-negative and this, then implies that condition (a) does not hold at all for $x \geq 0$. Then condition (b) will hold. Under this condition, the first factor comes out to be non-negative and remain non-negative after taking the summation over $i=1,2,\text{------}m_1$. Then, by (2.21) and by the finiteness of $E \| Gq \|$, the theorem is proved.

<u>Theorem 2.2</u>. Under the conditions of Theorem 2.1, either of the two sufficient conditions of the above theorem is also necessary for the program (2.1), (2.2) - (2.3) to be proper for given $h'x \geq 0$.

Note that if $x'Q^i x = 0$ for $i=1,2,\text{------},m_1$, the conditions theorem 2.1, reduce to the case of Dempster [36]. If $\|GMx\| = 0$, then these conditions become

(iii) $h + \sum_{i=1}^{m_1} Q^i v^i \geqslant 0,\ v^i \in R^n,\ v^i \geqslant 0,\ v^{i\prime} Q^i v^i \leq 1,\quad i=1,2,\text{------}m_1$

and

(iv) The $n\times n$ matrix $hh' + \sum_{i=1}^{m_1} Q^i$ is non-positive definite.

So we assume that $x'Q^i x \neq 0$ and $\|GMx\| \neq 0$.

Proof of Theorem 2.2. Let the program (2.1) - (2.3) be proper, i.e. $F(x)$ is finite and bounded below.

Now,

(2.22) $$F(x) = h'x + \sum_{i=1}^{m_1} (x'Q^i x)^{\frac{1}{2}} + E\ \|G(q-Mx)\|$$

$$\geqslant h'x + \sum_{i=1}^{m_1} (x'Q^i x)^{\frac{1}{2}} + E\ \|GMx\| - E\ \|Gq\|$$

Since $h'x \geqslant 0$, and $E\ \|Gq\| < \infty$, the right-handside of (2.22) is finite and

(2.23) $$h'x + \sum_{i=1}^{m_1} (x'Q^i x)^{\frac{1}{2}} + E\ \|GMx\| \geqslant 0.$$

This can be written as

(2.24) $$h'x + \sum_{i=1}^{m_1} \frac{x'Q^i x}{(x'Q^i x)^{\frac{1}{2}}} + E\left[\frac{x'M'G'GMx}{\|GMx\|}\right] \geqslant 0.$$

Taking $v^i = \dfrac{x}{(x'Q^i x)^{\frac{1}{2}}}$ and $u = \dfrac{GMx}{\|GMx\|}$, we see

that $v^i \geq 0$, $v^{i'}Q^i v^i = 1$, and $\|u\| = 1$.

Then (2.23) can be written as

$$h'x + \sum_{i=1}^{m_1} x'Q^i v^i - E(x'M'G'u) \geq 0 \quad \text{for all } x \geq 0.$$

This implies that

$$h + \sum_{i=1}^{m_1} Q^i v^i \geq E(M'G'u)$$

which is condition (i).

Also,

$$(2.25) \quad h'x + \sum_{i=1}^{m_1} (x'Q^i x)^{\frac{1}{2}} + E\|GMx\| \geq h'x + \sum_{i=1}^{m_1} (x'Q^i x)^{\frac{1}{2}} - E\ \|GMx\|$$

Left-handside of (2.25) is non-negative, the right-handside must satisfy

either

$$(a) \quad h'x + \sum_{i=1}^{m_1} (x'Q^i x)^{\frac{1}{2}} - E\ \|GMx\| \geq 0$$

or

$$(b) \quad h'x + \sum_{i=1}^{m_1} (x'Q^i x)^{\frac{1}{2}} - E\|GMx\| \leq 0.$$

If (a) is satisfied, then taking v^i as above and $u = \dfrac{GMx}{\|GMx\|}$, we obtain

the condition (i).

If (b) is satisfied, then, multiplying the relations (2.23) and (b), we get

$$0 \leq (E\|GMx\|)^2 - x'hh'x - \sum_{i=1}^{m_1} x'Q^i x - 2h'x \sum_{i=1}^{m_1} (x'Q^i x)^{\frac{1}{2}}$$

$$\leq E(x'M'G'GMx) - x'hh'x - \sum_{i=1}^{m_1} x'Q^i x$$

which implies that

$$E(M'GGM) - hh' - \sum_{i=1}^{m_1} Q^i$$

is an n x n non-negative definite matrix. This is the condition (ii). Hence, the Theorem is proved.

3. OTHER STOCHASTIC PROGRAMS

Other three types of stochastic programming models are discussed in this section. The decision equivalent deterministic programs for these problems are obtained, which come out to be quadratic type of programming problems. Under certain conditions the existence of finite values for the programs are discussed.

3.1. Stochastic Programming Problems with Quadratic Losses

We consider the following problem.

$$(3.2) \quad \underset{x \geqslant 0}{\text{Min}} \; E\left\{c'x + \underset{y \geqslant 0, z \geqslant 0}{\text{Min}} \left[d'y + f'z + \|Dz\|^2\right]\right\}$$

$$\text{subject to} \quad Ax \geqslant b,$$

$$P\left[Tx + Wy \geqslant a\right] \geqslant \alpha$$

$$P\left[Mx + Nz \geqslant q\right] = 1$$

which is the same problem considered in the previous section with $\|Dz\|$ replaced by its square. All the terms in this problem are as defined earlier.

To obtain the decision equivalent program for this problem we again follow the same procedure as applied for the problem (2.1) to (2.3). The expression (2.10) is the same for this program also. The expression (2.11) can be written as follows

$$(3.3) \quad \gamma(q-Mx) = \underset{z \geqslant 0}{\text{Min}} \left\{f'z + \|Dz\|^2 / Nz \geqslant q-Mx\right\}$$

$$= g'(q-Mx) + (q-Mx)'\, G'G(q-Mx) \quad , \text{say.}$$

The matrix $G'G$ is obviously a non-negative definite Program (3.2), then, can be expressed as follows

(3.4) $$\underset{x \geq 0}{\text{Min}} \; E\left\{c'x + \sum_{i=1}^{m_1} p_i(a_i^* - \bar{T}_i x + (x'V^i x)^{\frac{1}{2}}) + g'(q-Mx) + (q-Mx)'G'G(q-Mx)\right\}$$

subject to $Ax \geq b$.

Taking, $$h = E\left(c - \sum_{i=1}^{m_1} p_i \bar{T}_i - g'M - 2qG'GM\right)$$

and $S = E(M'G'GM)$, we obtain a decision equivalent deterministic program for (3.2) to be

(3.5) $$\underset{x \geq 0}{\text{Min}} \left\{h'x + \sum_{i=1}^{m_1} (x'Q^i x)^{\frac{1}{2}} + x'Sx = H(x)\right\}$$

subject to $Ax \geq b$.

The matrix S is non-negative definite, since

$$E(x'M'G'GMx) = x'E(M'G'GM)x \geq x'EM'G'GEMx \geq 0$$

by Jensen's inequality because $x'M'G'GMx$ is a convex function of the random variable Mx.

<u>Lemma 3.1.</u> The objective function $H(x)$ for the program (3.5) is convex over any convex subset of E^n. The Proof is obvious.

<u>Theorem 3.1</u> If the expectation of c and means and co-variances of q and M exist, then, for the program (3.2) to be proper the following condition is sufficient:

$\exists$ vectors $v^i \in E^n$, $v^i \geq 0$, $v^{i'}Q^i v^i \leq 1$, $i=1,2,$——m,

and $u \in E^\ell$, $\|u\| \leq \|Mx\|$, such that

$$(3.6) \qquad h + \sum_{i=1}^{m_1} Q^i v^i \geqslant E(M'G'u) .$$

If the lower bound of $H(x)$ is zero, then the above condition is also necessary.

Notice that the finiteness of the problem (3.2) also follows from the fact that $E\,\|q\|^2 < \infty \Rightarrow E\,\|q\| < \infty$

and $\quad E(x'M'G'GMx) \leq \|G\|^2 E\,\|M\|^2 \;\|x\|^2 < \infty \Rightarrow E\,\|MGx\| \leq \|G\|\,E\,\|M\|\,\|x\| < \infty$

and

$$\begin{aligned} E(qG'GMx) &\leq \|G\|^2 E(\,\|q\|\;\|M\|\,)\,\|x\| \\ &\leq \|G\|^2 (E\,\|q\|^2)^{\frac{1}{2}} (E\,\|M\|^2)^{\frac{1}{2}}\;\|x\| < \infty \end{aligned}$$

by Schwartz inequality, where norm $\|\;\|$ is an Euclidean vector norm.

<u>Proof of Theorem 3</u>. (Condition is sufficient). Let the condition (3.6) be satisfied. Then, for $x \geqslant 0$, we obtain that

$$h'x + \sum_{i=1}^{m_1} x'Q^i v^i \geqslant E(x'M'G'u)$$

Applying (2.16) and the given relation for v^i, we get

$$h'x + \sum_{i=1}^{m_1} (x'Q^i x)^{\frac{1}{2}} \geqslant E(x'M'G'u)$$

and then

$$\begin{aligned} h'x + \sum_{i=1}^{m_1} (x'Q^i x)^{\frac{1}{2}} + E(x'M'G'GMx) \\ \geqslant E(x'M'GGMx) + E(x'M'G'u). \end{aligned}$$

Now, since $\|u\| \leq \|MGx\| < \infty$, we have

$$|x'M'G'u| \leq \|MGx\| \ \|u\|$$
$$\leq \|MGx\|^2$$
$$= (x'M'G'GMx).$$

We, then, obtain that

$$(3.7) \quad h'x + \sum_{i=1}^{m_1} (x'Q^i x)^{\frac{1}{2}} + E(x'M'G'GMx) \geq 0$$

i.e. the function $H(x)$ is bounded below. Hence the sufficient part of the Theorem is proved.

(Condition is necessary) Let $H(x) \geq 0$. We assume, here, that $x'Q^i x \neq 0$, otherwise the condition will reduce to the form

$$h \geq E(M'G'u).$$

Then, since (3.7) is satisfied, it can be written as

$$(3.8) \quad h'x + \sum_{i=1}^{m_1} \left(\frac{x'Q^i x}{(x'Q^i x)^{\frac{1}{2}}} \right) + E(x'M'G'GMx) \geq 0.$$

Taking $v^i = \dfrac{x}{(x'Q^i x)^{\frac{1}{2}}}$ and $u = -GMx$, we see that

$v^i \geq 0$, and $v^{i'}Q^i v^i = 1$, and

$\|u\| = \|GMx\|$, and then, (3.8) becomes

$$h'x + \sum_{i=1}^{m_1} x'Q^i v^i \geq E(x'M'G'u) \quad \text{for all } x \geq 0$$

which reduces to the given condition (3.6).

Hence, the theorem is proved.

(3.9) Quadratic Programming Problems under risk with Linear Losses.

Consider the following problem:

(3.10) $$\min_{x \geq 0} E\left[c'x + x'Sx + \min_{y \geq 0} \{d'y\}\right]$$

subject to $Ax \geq b,$

$$P\left[Tx + Wy \geq a\right] \geq \alpha$$

where all terms but one are as defined in the previous section. S is a random matrix each element of which is a random variable having known distribution function. Matrix S is such that $E(S) = \bar{S}$ is a positive semi definite. T_i, $i=1,2,\text{——}m_1$, and a have known means and covariances.

The decision equivalent deterministic program can be obtained in the similar fashion as it is in the previous section. This deterministic program then, can be expressed as follows

(3.11) $$\min_{x \geq 0} \left\{H(x) = h'x + x'\bar{S}x + \sum_{i=1}^{m_1} (x'Q^i x)^{\frac{1}{2}}\right\}$$

subject to $Ax \geq b,$

where $\bar{S}$ is the expected value of S and $h = E\left[c - \sum_{i=1}^{m_1} p_i \bar{T}_i\right]$ and Q^i, $i=1,2,\text{——}m_1$ are as defined earlier.

Lemma 3.2. The function $H(x)$ is a convex function of x over the set $\{x/Ax \geq b, x \geq 0\}$.

Theorem 3.2. If the expected values of c and S and the means and covariances of T_i, $i=1,2,\text{——}m_1$ and a exist, then the program (3.10) is proper.

By lemma 3.2 and the conditions given, the proof is immediate.

(3.12) Linear Stochastic Programming Problems with Linear Losses and a Quadratic Constraint.

Consider the following problem:

$$(3.13)\quad \underset{x>0}{\text{Min}}\; E\left[\, c'x + \underset{y\geq 0}{\text{min}} \{d'y\} + \underset{s\geq 0}{\text{Min}} \{ks\}\right]$$

subject to $Ax \geq b$

$$P\left[T_i x + W_i y \geq a_i\right] \geq \alpha_i , \quad i = 1,2,\text{------}m_1 ,$$

$$\ell s + rx - (x'\Gamma x) \geq s$$

where c, A, b, T, W, a and α are as defined earlier

Γ is a positive semi-definite and symmetric matrix, r is n x 1 random vector, ℓ and s are scalars and s is a scalar variable. All the random variables involved in the problem i.e. r, c, T and a have known distributions with finite means and covariances.

The decision equivalent deterministic program for this problem can be obtained as

$$(3.14)\quad \underset{x\geq 0}{\text{Min}}\; E\left[h'x + \sum_{i=1}^{m_1} (x'Q^i x)^{\frac{1}{2}} + x'Sx\right]$$

subject to $Ax \geq b$,

where $h = c - \sum_{i=1}^{m_1} p_i \overline{T}_i - \frac{k}{\ell}\, s$

$S = \frac{k}{\ell}\Gamma$, k and ℓ being positive numbers.

This program is similar to the programs (3.11) and (3.5).

Program (3.14) is, also, a convex program.

Theorem 3.3. If the expected value of c and the means and covariances of r, T_i, and a are known to be finite, then the program (3.14) is proper.

From the programming models (3.5), (3.11) and (3.14) we see that such types of programming problems (may be of other types also) have their equivalent deterministic programs similar in nature and similar in form. Such deterministic programming models possess interesting and nice properties through which these programs can be solved efficiently.

4. SOME DUAL RELATIONS FOR THE DECISION EQUIVALENT PROGRAMS.

As it has been seen in the previous section that various types of stochastic programs are transformed into a single type of deterministic program. So, program like (3.5), (3.11) and (3.14) is very important in the respect that an optimal solution for this program will give optimal decisions for various stochastic programs. Though the program like (3.5) is very difficult to solve since the objective function may not be differentiable at some points in the region of consideration, but this can be solved with the help of duality theory, because, then, the dual program comes out to be concave quadratic program which can be solved easily. The computational aspects of the problem are not discussed here but a complete study of duality theory for this problem is given with the help taken from [37,48,49,54,55,107,108] .

We consider the following program to be called primal (P):

(4.1) $\text{Min } f(x) = h'x + \sum_{i=1}^{r} (x'Q^i x)^{\frac{1}{2}} + \frac{1}{2} x'Sx$

(4.2) subject to $Ax \geqslant b$

(4.3) $x \geqslant 0$

where h and x are $n \times 1$ and b is $m \times 1$ vector, S and Q^i, $i=1,2,\text{------}r$, are non-negative definite symmetric matrices and A is $m \times n$ matrix.

The dual of this programming problem denoted by (D) is expressed as follows

(4.4) $\quad$ Max $g(y,u,w^1,\text{------}w^r) = b'y - \frac{1}{2}\, u'Su$

(4.5) $\quad$ subject to $\quad A'y - \sum_{i=1}^{r} Q^i w^i - Su \leq h$

(4.6) $\quad w^{i'} Q^i w^i \leq 1, \; i=1,2,\text{------}r$

(4.7) $\quad y \geqslant 0$

where $u \in R^n$ and $w^i \in R^n$, $i=1,2,\text{------}r$.

We assume that the sets

$$C_P = \{x / Ax \geqslant b,\ x \geqslant 0\} \quad \text{and}$$

$$C_D = \left\{(y,u,w^1,\text{------}w^r) / A'y - \sum_{i=1}^{r} Q^i w^i - Su \leq h,\ w^{i'} Q^i w^i \leq 1,\ y \geqslant 0\right\}$$

are closed and bounded.* Either of the program (P) and (D) is called feasible if its constraint set is non-empty, otherwise called infeasible. A feasible solution for the program will be optimal if its objective function attains a finite extremum. We adopt the following convention that

$$\text{Inf } f(x) = +\infty \text{ if } C_P = \phi$$

and $\quad \sup g(y,u,w^1,\text{------}w^r) = -\infty$ if $C_D = \phi$

where ϕ denotes the null set.

---

* By this assumption we mean that those points for which C_P and C_D become unbounded, are excluded. This convention is carried throughout the Thesis where-ever it is assumed*

<u>Theorem 4.1</u>. (Weak Duality). For all feasible solutions to problems (P) and (D) respectively

$$\inf_{x \in C_P} f(x) \geq \sup_{(y,u,w,\text{---}w) \in C_D} g(y,u,w^1,\text{———}w^r)$$

<u>Proof</u>. If $C_P = \phi$ and $C_D = \phi$, the theorem is satisfied obviously under the above hypotheses. So, we assume that $C_P \neq \phi$ and $C_D \neq \phi$.

Then, both problems become feasible. Now, for all $x \in C_P$, we have

$$f(x) = h'x + \sum_{i=1}^{r} (x'Q^i x)^{\frac{1}{2}} + \tfrac{1}{2}\, x'Sx$$

$$\geq h'x + \sum_{i=1}^{r} (x'Q^i x)^{\frac{1}{2}} (w^{i'} Q^i w^i)^{\frac{1}{2}} + \tfrac{1}{2}\, x'Sx$$

(by (4.6))

$$\geq h'x + \sum_{i=1}^{r} x'Q^i w^i + x'Su - \tfrac{1}{2}\, u'Su$$

(by (2.16) and (4.8) below)

$$\geq y'Ax - \tfrac{1}{2}\, u'Su \quad \text{(by (4.5) and (4.3))}$$

$$\geq b'y - \tfrac{1}{2}\, u'Su \quad \text{(by (4.2) and (4.7))}$$

$$= g(y,u,w^1,\text{———}w^r) \text{ for all } (y,u,w^1,\text{———}w^r) \in C_D.$$

Thus, the theorem is proved.

We have used above the following well known relation

(4.8) $\tfrac{1}{2}\, x'Sx + \tfrac{1}{2}\, u'Su \geq u'Sx.$

Let x^* be an optimal solution of the problem (P). Then, consider the following program denoted by (P^1).

$$\text{Min } F(x) = h'x + x^{0'}Sx - \tfrac{1}{2}\, x^{0'}Sx^0 + \sum_{i=1}^{r} (x'Q^i x)^{\frac{1}{2}}$$

subject to $\quad Ax \geqslant b,$

$$x \geqslant 0$$

The set of constraints for the problem (P^1) is also C_P.

<u>Theorem 4.2</u>. The solution x^0 of the problem (P) is optimal if and only if x^0 is an optimal solution of the problem (P^1).

<u>Proof</u>. Suppose that there exist another solution x^* to the problem (P^1) such that

$$F(x^*) < F(x^0) \tag{4.9}$$

where x^0 is an optimal solution of (P).

Let $0 < \lambda < 1$, then $x_\lambda = x^0 + \lambda(x^* - x^0) \in C_P$ i.e. x_λ is feasible for (P) and (P^1) both.

Then

$$f(x_\lambda) = f(x^0 + \lambda(x^* - x^0))$$

$$= h'x_\lambda + \tfrac{1}{2}\, x'_\lambda\, Sx_\lambda + \sum_{i=1}^{r}\Big[\lambda^2 x^{*'}Q^i x^* + 2\lambda(1-\lambda)x^{*'}Q^i x^0 + (1-\lambda)^2 x^{0'}Q^i x^0\Big]^{1/2}$$

$$\leq h'x_\lambda + \tfrac{1}{2} x'_\lambda\, Sx_\lambda + \sum_{i=1}^{r}\Big[\lambda(x^{*'}Q^i x^*)^{\frac{1}{2}} + (1-\lambda)\,(x^{0'}Q^i x^0)^{\frac{1}{2}}\Big]$$

Thus,

$$f(x_\lambda) - f(x^0) \leq \lambda\,(h' + x^{0'}S)\,(x^* - x^0) + \tfrac{1}{2}\,\lambda^2 (x^* - x^0)'\, S\,(x^* - x^0)$$

$$+ \lambda \sum_{i=1}^{r}\Big[(x^{*'}Q^i x^*)^{\frac{1}{2}} - (x^{0'}Q^i x^0)^{\frac{1}{2}}\Big]$$

$$= \lambda\big[F(x^*) - F(x^0)\big] + \tfrac{1}{2}\,\lambda^2 (x^* - x^0)'\, S(x^* - x^0). \tag{4.10}$$

Since, the first term in the right-hand side of (4.10) is strictly negative and is independent of λ, choosing λ sufficiently small, we can obtain the value at the right-hand-side of (4 10) which is strictly negative. This implies that

(4.11) $$f(x_\lambda) - f(x^0) < 0$$

which contradicts the hypothesis that x^0 is an optimal solution of the problem (P). Thus, the assumption (4.9) is invalid and x^0 is an optimal solution of (P^1)

Conversely, let x^0 be an optimal solution of (P^1). Then, for all $x \in C_p$,

$$F(x^0) - F(x) = (h' + x^{0\prime}S)(x^0 - x) + \sum_{i=1}^{r} \left[(x^{0\prime}Q^i x^0)^{\frac{1}{2}} - (x'Q^i x)^{\frac{1}{2}} \right] \leq 0$$

Now, for all $x \in C_p$,

$$\begin{aligned} f(x^0) - f(x) &= h'(x^0 - x) + \tfrac{1}{2} x^{0\prime}Sx^0 - \tfrac{1}{2} x'Sx \\ &\quad + \sum_{i=1}^{r} \left[(x^{0\prime}Q^i x^0)^{\frac{1}{2}} - (x'Q^i x)^{1/2} \right] \\ &\leq h'(x^0 - x) + x^{0\prime} S(x^0 - x) \\ &\quad + \sum_{i=1}^{r} \left[(x^{0\prime}Q^i x^0)^{\frac{1}{2}} - (x'Q^i x)^{\frac{1}{2}} \right] \\ &\qquad \text{(by (4.8))} \\ &= F(x^0) - F(x) \\ &\leq 0 \end{aligned}$$

This implies that x^0 is an optimal solution of (P).

Hence, the theorem is proved.

<u>Theorem 4.3</u>. If $x = x^0$ is a minimizing solution of the problem (P), then the maximizing solution exists for the problem (D) and the optimal value μ in (D) is x^0, and the optimal values of the two problems are equal.

Proof. Let $x = x^0 \in C_p$ be a minimizing solution of the problem (P). Then, by Theorem (4.2), the problem (P^1) has an optimal solution x^0 and conversely.

By the duality theorem for homogeneous programming [49,107] the dual of the program

$$(4.12)\quad \underset{x \in C_p}{\text{Min}}\ F'(x) = h'x + x^{0\prime}Sx + \sum_{i=1}^{r} (x'Q^i x)^{\frac{1}{2}}$$

is given by

$$(4.13)\quad \text{Max}\ \left\{ b'y = G'(y, x^0, w^1, \text{---} w^r) \right\}$$

subject to

$$A'y \leq \sum_{i=1}^{r} Q^i w^i + Sx^0 + h$$

$$w^{i\prime} Q^i w^i \leq 1,\ i = 1, 2, \text{---} r$$

$$y \geqslant 0$$

such that

$$(4.14)\ \text{Max}\ G(y, x^0 w^1, \text{---} w^r) = \text{Min}\ F'(x).$$

By (4.13) and (4.14), the dual of (P') can be expressed as follows

$$\text{Maximize}\ G(y, x^0, w^1, \text{---} w^r) = b'y - \tfrac{1}{2}\, x^{0\prime}\, Sx^0$$

$$(4.15)\ \text{subject to}\quad A'y \leq \sum_{i=1}^{r} Q^i w^i + Sx^0 + h$$

$$w^{i\prime} Q^i w^i \leq 1,\ i = 1, 2, \text{---} r$$

$$y \geqslant 0$$

such that

$$\text{Max. } G(y, x^{o}, w^{1}, \text{———} w^{r}) = \text{Min. } F(x)$$

So, if $(y^{o}, x^{o}, w_{o}^{1}, \text{———} w_{o}^{r})$ is an optimal solution of the problem (4.15), one obtains

$$(4.16)\quad h'x^{o} + \tfrac{1}{2} x^{o'} Sx^{o} + \sum_{i=1}^{r} (x^{o'} Q^{i} x^{o})^{\frac{1}{2}} = b'y^{o} - \tfrac{1}{2} x^{o'} Sx^{o}.$$

We also, see that $(y^{o}, x^{o}, w_{o}^{1}, \text{———} w_{o}^{r})$ belongs to the set C_D, i.e. it is feasible for the program (D). Now, let $(y, u, w^{1}, \text{———} w^{r}) \in C_D$ be any other feasible solution for (D). Then,

$$g(y^{o}, x^{o}, w_{o}^{1}, \text{———} w_{o}^{r}) - g(y, u, w^{1}, \text{———} w^{r})$$

$$= b'(y^{o} - y) - \tfrac{1}{2} x^{o'} Sx^{o} + \tfrac{1}{2} u'Su$$

$$\geqslant b'(y^{o} - y) + x^{o'} Su - x^{o'} Sx^{o}$$

(by (4.8))

$$= h'x^{o} + x^{o'} Sx^{o} + \sum_{i=1}^{r} (x^{o'} Q^{i} x^{o})^{\frac{1}{2}} - b'y + x^{o'} Su - x^{o'} Sx^{o}$$

(by(4.16))

$$\geqslant h'x^{o} + \sum_{i=1}^{r} (x^{o'} Q^{i} x^{o})^{\frac{1}{2}} (w^{i'} Q^{i} w^{i})^{\frac{1}{2}} - b'y + x^{o'} Su$$

(by(4.6))

$$\geqslant h'x^{o} + \sum_{i=1}^{r} x^{o'} Q^{i} w^{i} - b'y + x^{o'} Su$$

(by (2.16))

$$\geqslant y'Ax^{o} - b'y \quad \text{(by (4.5) and since } x^{o} \in C_P)$$

$$\geqslant 0 \quad \text{(by (4.7) and since } x^{o} \in C_P) .$$

This implies that $(y^o, x^o, w_o^1, \text{------} w_o^r)$ is a maximizing solution of the problem (D) which shows that x^o is the optimal value of u in (D). Also, by (4.16), the optimal values of the two problems (P) and (D) are equal. Thus, the theorem is proved.

Theorem 4.4. The set $C_P \neq \phi$ if and only if $C_D \neq \phi$.

Proof. Let $C_P \neq \phi$. Then, there exists a solution $x^o \in C_P$. Now, if possible, let $C_D = \phi$. It, then follows that there exists no point $(y, u, w^1, \text{------} w^r) \in C_D$ such that

$$(4.17) \qquad (A', -G, S) \begin{pmatrix} y \\ \eta \\ \zeta \end{pmatrix} \leq h,\ y \geq 0,\ \eta \geq 0,\ \zeta \geq 0.$$

Otherwise, by taking $w^i = 0$, $i = 1, 2, \text{------} r$, we see that $(y, u, 0, \text{------} 0) \in C_D$, a contradiction to its voidness. Now, since the system (4.17) has no solution, by the usual feasibility theorem for linear inequalities [54,55] the following system

$$(4.18) \qquad Ax \geq 0,\ Sx = 0,\ x \geq 0,\ h'x < 0$$

has a solution. Thus, for any $k > 0$, $x^o + kx \in C_P$ which implies that C_P is unbounded, a contradiction to the boundedness of C_P. Hence, $C_D \neq \phi$.

Conversely, let $C_D \neq \phi$. Then, $\exists$ a solution $(y^o, u^o, w_o^1, \text{------} w_o^r) \in C_D$. If possible, let $C_P = \phi$. This implies that the system (4.2), (4.3) has no solution, otherwise it will contradict the voidness of C_P. Then, by the feasibility theorem for linear inequalities [54,55] the system

$$A'y \leq 0,\ b'y > 0,\ y \geq 0$$

must have a solution. And consequently, for any $k > 0$, $(y^o + ky, u^o, w_o^1, \text{---} w_o^r) \in C_D \Rightarrow$ that C_D

is unbounded $\Rightarrow$ a contradiction to the hypothesis that C_D is bounded. Hence $C_p \neq \phi$. Thus, the theorem is proved

<u>Corollary</u>: $C_p = \phi \Leftrightarrow C_D = \phi$.

Proof is immediate by Theorem 4.4.

Theorem 4.5 (Converse of Theorem 4.3) If $(y^o, u^o, w_o^1, \text{———} w_o^r)$ is an optimal solution of the dual program (D), then $\exists\, x^o = u^o$ such that x^o is an optimal solution to the problem (P) and the respective extremal values of the programs are equal.

<u>Proof</u>. Let $(y^o, u^o, w_o^1, \text{———} w_o^r) \in C_D$ be an optimal solution to the dual (D). Let $u = \eta - \xi$, $\eta \geq 0$, $\xi \geq 0$. Then, (D) can be rewritten as (D$'$):

$$\text{Maximize} \quad \begin{pmatrix} b \\ o \\ o \end{pmatrix}' \begin{pmatrix} y \\ \eta \\ \xi \end{pmatrix} - \tfrac{1}{2} \begin{pmatrix} y \\ \eta \\ \xi \end{pmatrix}' \begin{bmatrix} o & o & o \\ o & S & -S \\ o & -S & S \end{bmatrix} \begin{pmatrix} y \\ \eta \\ \xi \end{pmatrix} = \beta' z - \tfrac{1}{2} z' \Gamma z$$

subject to

$$(4.19) \quad B'z = (A\ ,\ -S,\ S) \begin{pmatrix} y \\ \eta \\ \xi \end{pmatrix} \leq h + \sum_{i=1}^{r} Q^i w^i$$

$$w^{i'} Q^i w^i \leq 1, \quad i=1,2, \text{———} r$$

$$y \geq 0, \quad \eta \geq 0, \quad \xi \geq 0, \quad z \geq 0$$

where $(b', o, o) = \beta'$, $(y', \eta' \xi') = z'$ and $\Gamma = \begin{bmatrix} o & o & o \\ o & S & -S \\ o & -S & S \end{bmatrix}$ and $B' = (A', -S, S)$.

Since $(y^o, u^o, w_o, \text{———} w_o^r)$ is optimal solution for (D), it implies that

$(y^o,\ z^o\ ,\ w_o^1, \text{———} w_o^r)$ is an optimal solution for (D$'$), where

$u^o = \eta^o - \xi^o$, $\eta^o \geq 0$, $\xi^o \geq 0$ and $z^o = \begin{pmatrix} y^o \\ \eta^o \\ \xi^o \end{pmatrix}$.

Now consider the following linear program

(4.20) Maximise $\beta' z - z' \Gamma z^0 + \frac{1}{2} z^{0'} \Gamma z^0$

subject to $B'z \leq h + \sum_{i=1}^{r} Q^i w^i$

$w^{i'} Q^i w^i \leq 1, \; i=1,2,\text{------}r$

$z \geqslant 0.$

Similar to the proof of Theorem 4.2, we can show that $(z^0, w_0^1, \text{------} w_0^r)$ is an optimal solution to the problem (4.19) $\Longleftrightarrow$ $(z^0, w_0^1, \text{------} w_0^r)$ is an optimal solution to (4.20).

Consider the following homogeneous program

(4.21) Maximise $\left\{ \beta' z - z' \Gamma z^0 = \theta_1(z, w^1, \text{------} w^r) \right\}$

subject to

$B'z \leq h + \sum_{i=1}^{r} Q^i w^i$

$w^{i'} Q^i w^i \leq 1, \; i=1,2,\text{-----}r$

$z \geqslant 0.$

Then, the programs (4.20) and (4.21) will have the same optimal solution.

To the dual of (4.21) we can write as

(4.22) Minimise $F_1(x) = h'x + \sum_{i=1}^{r} (x' Q^i x)^{\frac{1}{2}}$

subject to $Bx + \Gamma z^0 \geqslant \beta$

$x \geqslant 0$

Now, we show that the program (4.21) is equivalent to the program

(4.23) Maximise $\theta_1(z, w^1, \text{------} w^r)$

subject to $x'B'z \leq F_1(x)$

$z \geqslant 0.$

Let $(z, w^1, \text{———} w^r)$ be any feasible solution to (4.21). Then, for all $x \geq 0$,

$$x'B'z \leq h'x + \sum_{i=1}^{r} x'Q^i w^i$$

$$\leq h'x + \sum_{i=1}^{r} (x'Q^i x)^{\frac{1}{2}}$$

$$= F_1(x).$$

Thus z is a feasible for (4.23). Conversely if (z_1) is any feasible solution to (4.23), then, for all $x \geq 0$

$$x'B'z_1 \leq h'x + \sum_{i=1}^{r} (x'Q^i x)^{\frac{1}{2}}$$

which means that

$$-Ix \leq 0 \Longrightarrow \left[z_1'B - h'\right] \; x \leq \sum_{i=1}^{r} (x'Q^i x)^{\frac{1}{2}} \qquad \text{for all } x \geq 0.$$

where I is an identity matrix of appropriate dimension.

It, then, follows from [108,48] that there exists

$$(\xi, w^1, \text{———} w^r) \text{ such that}$$

$$-I\xi + \sum_{i=1}^{r} Q^i w^i = B'z_1 - h$$

$$w^{i'} Q^i w^i \leq 1, \; i=1,2, \text{———} r$$

$$\xi \geq 0$$

which implies that

$$B'z_1 \leq h + \sum_{i=1}^{r} Q^i w^i$$

$$w^{i'} Q^i w^i \leq 1, \; i=1,2, \text{———} r$$

$$z_1 \geq 0$$

i.e. $(z_1, w^1, \text{-----} w^r)$ is feasible solution to (4 21).

Similarly, it can be shown that the program (4.22) is equivalent to

$$(4.24) \quad \text{Min} \quad F_1(x) = h'x + \sum_{i=1}^{r} (x'Q^i x)^{\frac{1}{2}}$$

$$\text{subject to} \quad z'Bx \geqslant G_1(z, w^1, \text{-----} w^r)$$

$$x \geqslant 0.$$

Since, the problem (D) has an optimal solution, by Theorem 4.4, the problem (P) must have a feasible solution, and since C_p is bounded,

$$(4.25) \quad Bx \geqslant 0,\ F_1(x) \leq 0,\ x \geqslant 0 \Longrightarrow x = 0.$$

Also, $F_1(x)$, $G_1(z, w^1, \text{-----} w^r)$ are linear, positively homogeneous continuous convex and concave functions respectively. Then, by [49], problems (4.23) and (4.24) will have their optimal solutions and their extreme values will be equal. It, then, follows that programs (4.21) and (4.22) will have their optimal solutions and their extreme values will be equal. Thus, if x^o is optimal for (4.22), then we have

$$(4.26) \quad \beta' z^o - z^{o'} \Gamma z^o = h'x^o + \sum_{i=1}^{r} (x^{o'} Q^i x^o)^{\frac{1}{2}}$$

Now, substituting the values of β, Γ and z^o, we obtain that

$$(4.27) \quad \begin{pmatrix} b \\ o \\ o \end{pmatrix}' \begin{pmatrix} y^o \\ \eta^o \\ \xi^o \end{pmatrix} - \begin{pmatrix} y^o \\ \eta^o \\ \xi^o \end{pmatrix}' \begin{bmatrix} o & o & o \\ o & S & -S \\ o & -S & S \end{bmatrix} \begin{pmatrix} y^o \\ \eta^o \\ \xi^o \end{pmatrix} = h'x^o \sum_{i=1}^{r} (x'Q^i x^o)^{\frac{1}{2}}$$

and $Sx^o = S(\eta^o - \xi^o) = Su^o$.

Then, (4.27) reduces to

$$(4.28) \quad b'y^o - x^{o'} Sx^o = h'x^o + \sum_{i=1}^{r} (x^{o'} Q^i x^o)^{\frac{1}{2}}$$

and $x^o = \eta^o - \xi^o = u^o$ for the optimal.

By Theorem 4.1, (4.28) implies that x^o is an optimal solution to the problem (P). Hence the theorem is proved.

CHAPTER - V

TWO-STAGE NONLINEAR PROGRAMMING PROBLEMS UNDER RISK AND UNCERTAINTY

INTRODUCTION

Many practical problems of the real world can be expressed mathematically in terms of two-stage programming problems under risk and uncertainty. One such example is that of the production inventory problem where the demand and the cost vectors can only be specified in advance by their distributions. A decision vector may represent the amount to be produced and/ or bought from the flustuating market before the actual demand and the costs of producing items and/ or costs at the flustuating market are known. This is the first stage decision and results in production of or purchase of some inventory at some cost. Whenever the random variables are observed, the second stage decision is taken to compensate the inaccuracies obtained in the previous

decision at the minimum cost. The purpose remains to find the total minimum expected cost.

This chapter is divided into two parts, Part I consists of the discussion about the set of feasible solutions, second-stage program, the equivalent deterministic program and the optimality conditions for the two stage non-linear convex programming under uncertainty. The set of feasible solution (assumed to be not null), is shown to be convex. The deterministic equivalent program comes out to be convex. Since no convexity conditions for the functions is applied in the proof of the optimality results for the second stage and equivalent deterministic programs, they may hold for the general non-convex functions also, say, for example Theorem 5.1 also holds for Pseudo-convex function.

A study of the problem considered in part I with additional linear probabilistic constraints is made in part II. The permanently feasible solution set and the equivalent deterministic program turn out to be convex. A study of dual relations for the equivalent deterministic program having the objective function as the sum of the differentiable and non-differentiable functions is performed.

PART - I

1. Problem Statement

A general two-stage stochastic programming problem can be expressed as follows:

$$\underset{x}{\text{Minimise}} \quad E\left[\Phi(c,x) + \min_{y}\left\{\psi(d,y)\right\}\right]$$

$$(1.1) \quad \text{subject to} \quad f(x) \geqslant 0$$

$$P\left[g(x) + h(y) \geqslant b\right] = 1$$

where c,d,x,y and b are $k \times 1$, $\ell \times 1$, $n \times 1$, $\bar{n} \times 1$ and $\bar{m} \times 1$ vectors respectively, ϕ and ψ are scalar functions and f,g and h are vector-valued functions of their respective arguments. The coordinates of c,d and b are random variables with known distribution functions, E denotes the expectation and P is the probability. 1 is $\bar{m} \times 1$ sum vector. Equalities and inequalities hold componentwise.

It is assumed that the functions ϕ, ψ and each component of -g, -h and -f are convex functions of their respective arguments. It is further assumed that the joint probability space $(\Omega, \mathcal{F}, P_r)$ is given, where Ω is a Borel subset of R^N, $N = k + \ell + \bar{m}$, $\mathcal{F}$ is a σ- field on Ω which include Borel subsets and P_r is the probability measure defined on $\mathcal{F}$. The coordinates of vectors c,d and b are the components of the vector $p' = (c', d', b')$ with an associated joint distribution function F. The marginal probability spaces corresponding to c,d and b are denoted by Ω_c, Ω_d and Ω_b respectively. The random variables c, d and b are assumed independent i.e. $\Omega = \Omega_c \times \Omega_d \times \Omega_b$. Expected values and the higher moments of p are assumed to exist. We think of Ω as a convex set with measure one, if it is not convex we can replace it by its convex hull and fill up $\mathcal{F}$ by sets of measure zero.

All quantities considered are reals. It is assumed that the programming problem (1.1) is proper i.e. attains finite value.

2. THE SET OF FEASIBLE SOLUTIONS:

A vector x is said to be a feasible solution to the problem (1.1) if it satisfies the first stage constraints and is such that for all $p(w) \in W$ where $p: \Omega \to W$, the second stage program

$$(2.1)\quad \min_{y} \ \psi(d,y)$$

$$\text{subject to} \quad h(y) \geqslant b-g(x)$$

is feasible. A solution x to the problem (1.1) is said to be 'permenently feasible' if the constraints are satisfied with probability one.

Let $K_1 = \{ x/f(x) \geqslant 0 \}$

Since each component of $f(x)$ is a concave function of x, K_1 is a convex set.

Let $K_{2p(w)} = \{ x/ \ \exists y \ \text{ such that } h(y) \geqslant b-g(x) \}$.

<u>Lemma 2.1</u>. For fixed p, the set $K_{2p(w)}$ is convex.

Proof is immediate, since, components of g and h are concave functions.

Let $K_2 = \bigcap_{w \in \Omega} K_{2p(w)}$

$$= \{ x/ \forall \ p(w) \in W, \ \exists y \text{ such that } \ h(y) \geqslant b-g(x) \}$$

<u>Lemma 2.2.</u> The set K_2 is convex.

Since the intersection of convex sets is convex, the proof follows immediately.

<u>Lemma 2.3</u>. The set $K = K_1 \bigcap K_2$ is convex.

Now, consider the multi-stage nonlinear programming under uncertainty of the form

$$\min_{x} \ E \left[\phi(c,x) + \min_{y} \ \{ \psi(d,y) \} \right]$$

$$(2.2) \quad \text{subject to} \quad f(x) \geqslant 0$$

$$g(x) + h(y) \geqslant b$$

$$y \in U$$

where U is a convex set and all other terms are as defined earlier.

Lemma 2.4. The solution set K of the problem (2.2) is convex.

Proof. Since U is a convex set, the set $H(U) = \{t / h(y) \geq t\}$ is convex and so $b - H(U)$ is convex. Since each convex set is translate by b of a given convex set, these are parallel, and so intersection is also a convex set. Hence the Lemma follows.

3. THE SECOND STAGE PROGRAM

For given b and x, we consider the following program called the second-stage program.

$$\begin{array}{lll} & \text{Minimise} & \psi(d,y) \\ (3.1) & \text{subject to} & h(y) \geq b - g(x) \\ & & y \in R^{\bar{n}} \end{array}$$

Let there exist functions $\pi(b,x) \geq 0$ such that we can define a function $\gamma : R^{\bar{n}} \times V \longrightarrow R$ as

$$\gamma(y, \pi(b,x)) = \psi(d,y) + \langle \pi(b,x),\ b - g(x) - h(y) \rangle$$

where $V = \{\pi(b,x) / \ \pi(b,x) \geq 0,\ \gamma(y, \pi(b,x))$ attains its minimum over $R^{\bar{n}}\} \subset R^{\bar{m}}$.

Now, define a function $\delta : V \longrightarrow R$ as follows

$$\delta(\pi(b,x)) = \min_{y \in R^{\bar{n}}} \gamma(y, \pi(b,x))$$
$$= \gamma(y^*, \pi(b,x))$$

Since b and x are given, we shall write π for $\pi(b,x)$ and $\gamma(y, \pi)$ for $\gamma(y, \pi(b,x))$ respectively unless otherwise specified.

Lemma 3.1. The function $\delta(\pi)$ is concave over the convex subsets of V.

Proof is similar to given in [5].

Since $\delta(\pi)$ is concave over the convex subsets of V, it is continuous in the interior of the convex subsets of V.

If $\delta(\pi)$ is differentiable at $\pi \in V$, then

$$\nabla \delta(\pi) = b - g(x) - h(y)$$

the proof of which can be seen in [53].

<u>Theorem 3.1</u>. For given b and x, let π^* be a maximising point for $\delta(\pi)$ which is differentiable at π^*. Then, any point $y^* \in R^{\bar{n}}$, for which

$$\delta(\pi) = \min_{y \in R^{\bar{n}}} \gamma(y, \pi) = \gamma(y^*, \pi)$$

holds, is a solution of the problem (3.1). Furthermore,

$$\psi(d, y^*) = \delta(\pi^*) .$$

<u>Proof</u>. Since $\delta(\pi)$ is differentiable at $\pi^* \in V$ and π^* is a maximising point for $\delta(\pi)$, it implies that

$$\nabla \delta(\pi^*) = b - g(x) - h(y^*) = 0.$$

Then,

$$\delta(\pi^*) = \min_{y \in R^{\bar{n}}} \gamma(y, \pi^*) = \min_{y \in R^{\bar{n}}} \psi(d, y) + \langle \pi^*, b - g(x) - h(y) \rangle$$

$$= \psi(d, y^*) + \langle \pi^*, b - g(x) - h(y^*) \rangle = \psi(d, y^*).$$

Now,

$$(3.2) \qquad \delta(\pi^*) = \min_{y \in R^{\bar{n}}} \gamma(y, \pi^*) \leq \gamma(y, \pi^*) \text{ for all } y \in R^{\bar{n}}$$

Also, since $\pi^* \geqslant 0$, and $b - g(x) - h(y) \leq 0$,

we have,

$$(3.3) \qquad \langle \pi^*, b - g(x) - h(y) \rangle \leq 0$$

By (3.2), we see that

$$\delta(\pi^*) = \psi(d, y^*) \leq \psi(d, y) + \langle \pi^*, b - g(x) - h(y) \rangle$$

$$\leq \psi(d, y) \qquad \text{(by (3.3)), for all } y \in R^{\bar{n}}$$

which implies that y^* is an optimal for the program (3.1).

Theorem 3.2. The existence of $\pi^* \in V$ maximizing $\delta(\pi)$ implies and is implied by the existence of a $y^* \in R^{\bar{n}}$ such that (y^*, π^*) is a saddle point for the function $\gamma(y, \pi)$. Conversely, if (y^*, π^*) is a saddle point for the function $\gamma(y, \pi)$, then y^* is optimal for the program (3.1) and π^* is a maximizing point for $\delta(\pi)$ with

$$\delta(\pi) = \min_{y \in R^{\bar{n}}} \gamma(y, \pi) = \gamma(y^*, \pi) \text{ and}$$

$$\delta(\pi^*) = \min_{y \in R^{\bar{n}}} \gamma(y, \pi^*) = \gamma(y^*, \pi^*) \tag{3.4}$$

and $\pi^* \in V$ and y^* being feasible for (3.1).

Proof. Since $\pi^* \in V$ and π^* maximizes $\delta(\pi)$, we have

$$\delta(\pi^*) \geq \delta(\pi) \quad \text{for all } \pi \in V.$$

Thus,

$$\gamma(y^*, \pi^*) = \min_{y \in R^{\bar{n}}} \gamma(y, \pi^*) = \delta(\pi^*) \geq \delta(\pi)$$

$$= \min_{y \in R^{\bar{n}}} \gamma(y, \pi) = \gamma(y^*, \pi) \quad \text{for all } \pi \in V. \tag{3.5}$$

Also,

$$\delta(\pi^*) = \gamma(y^*, \pi^*) = \min_{y \in R^{\bar{n}}} \gamma(y, \pi^*) \leq \gamma(y, \pi^*) \text{ for all } y \in R^{\bar{n}}. \tag{3.6}$$

Thus, from (3.5) and (3.6) we obtain

$$\gamma(y^*, \pi) \leq \gamma(y^*, \pi^*) \leq \gamma(y, \pi^*). \tag{3.7}$$

Conversely, since (3.7) is satisfied and (3.4) holds, we have

$$\delta(\pi) = \gamma(y^*, \pi) \leq \gamma(y^*, \pi^*) = \delta(\pi^*) \text{ for all } \pi \in V,$$

which implies that

$$\delta(\pi) \leq \delta(\pi^*) \quad \text{for all } \pi \in V,$$

i.e. π^* is a maximizing point for $\delta(\pi)$.

ow, from first inequality of (3.7) we have

$$\langle \pi , b-g(x)-h(y^*)\rangle \leq \langle \pi^*, b-g(x+-h(y^*)\rangle$$

for all $\pi \in V$,

n particular for $\pi = 0$, we get

$$(3.8) \qquad \langle \pi^*, b-g(x)-h(y^*)\rangle \geq 0$$

But y^* is feasible for the program (3.1), and since $\pi^* \in V$ $\pi^* \geq 0$,

we have

$$(3.9) \qquad \langle \pi^x, b-g(x) - h(y^*)\rangle \leq 0.$$

From, (3.9) and (3.8) we obtain

$$(3.10) \qquad \langle \pi^*, b-g(x) - h(y^*)\rangle = 0.$$

Then, from the second inequality of (3.7) with (3.10) we get

$$\Psi(d,y^*) \leq \Psi(d,y) + \langle \pi^*, b-g(x)-h(y)\rangle$$

$$\leq \Psi(d,y) \qquad \text{for all } y \in R^{\bar{n}}, \text{ and all } y \text{ feasible for } (3.1)$$

This implies that y^* is optimal point of (3.1).

Hence, the theorem is proved.

4. A DECISION EQUIVALENT DETERMINISTIC PROGRAM

A deterministic program $\min_{x \in K} H(x)$ is said to be decision equivalent to the program (1.1) if the set K of feasible solutions x's of (1.1) is the same as that of this program and the optimal values of the two programs are identical.

The second stage program is considered in the previous section. Let

$$(4.1) \qquad P(x,p) = \left\{ \min_{y} \Psi(d,y)/h(y) \geq b-g(x) \right\} .$$

emma 4.1. The function $P(x,p)$ is convex in x on K_2.
he proof is similar to as given in [80].

Now, by the theory of duality in nonlinear programming 66,81,82,153], the dual program for (4.1) can be written as follows:

$$(4.2) \quad Q(x,p) = \left\{ \max_{y,\pi} \ \nu(y,\pi) / \nabla\nu(y,\pi) = 0, \ \pi \geq 0 \right\}$$

$$= P(x,p).$$

Lemma 4.2. The function $Q(x,p)$ is convex in x on K_2.
Proof follows from Theorem 1 Chapter II.

Let $EQ(x,p) = Q(x)$.

Lemma 4.3. The function $Q(x)$ is convex in x on K_2.

Lemma 4.4. If the function ψ, and each component of $-g$, and $-h$ are convex and continuous in the domain of consideration, then the function $Q(x)$ is convex and continuous on K_2.

Since $Q(x)$ is convex, on K_2 it is continuous in the interior of K_2. Similarly $Q(x,p)$ is continuous in the interior of K_2. The proof for the continuity of $Q(x,p)$ follows from [80].

Now,

$$Q(x) = \int_\Omega \lim_{i\to\infty} Q(x_i, p(w))\, dP_r(w) = \int_\Omega Q(x, p(\omega))\, dP_r(\omega)$$

$$= \lim_{i\to\infty} \int_\Omega Q(x_i, p(w))\, dP_r(w)$$

where $\{x_i\}$ is a sequence in K_2 converging to a boundary point x on K_2.

(4.3) implies that $Q(x)$ isalso continuous on K_2.

The equivalent deterministic program for the problem (1.1) then becomes

Conversely, if x^* is feasible for (1.1) and (y^*, π^*) is feasible or (4.2), then, under the relation (5.2) x^* is optimal for the program 1.1) and $(y^*, \pi^*/x^*)$ is optimal for the program (4.2).

Proof. Since, $\gamma(y^*, \pi/x^*) \leq \gamma(y^*, \pi^*/x^*)$,

one has

$$\theta(x^*, y^*, \pi) = \phi(x^*) + E\,(y^*, \pi/x^*)$$

(5.3) $$\leq \phi(x^*) + E\,\gamma(y^*, \pi^*/x^*)$$

$$= \theta(x^*, y^*, \pi^*).$$

Now, let $\theta(x, y^*, \pi^*) = \phi(x) + E\,\gamma(y^*, \pi^*/x)$

Then, by the optimality of x^*, we have

(5.4) $$\theta(x, y^*, \pi^*) \geq \phi(x^*) + E\,\gamma(y^*, \pi^*/x^*) = \theta(x^*, y^*, \pi^*).$$

From (5.3) and (5.4), the relation (5.2) is satisfied.

To establish the converse, let $x^* \in K$, by (5.4) x^* comes out to be optimal for the program (1.1). Let (y^*, π^*) be feasible for the program (4.2).

To show that (y^*, π^*) is optimal for (4.2), we assume the converse.

Suppose that (y^*, π^*) is not optimal for (4.2). Let $(y^*, \bar{\pi})$ be an optimal for the program (4.2). Then, the following holds

$$\gamma(y^*, \pi^*/x^*) \leq \gamma(y^*, \bar{\pi}/x^*) \leq \gamma(y, \bar{\pi}/x^*).$$

But $(y^*, \bar{\pi})$ is optimal for (4.2), we have

$$\gamma(y^*, \pi^*/x^*) < \gamma(y^*, \bar{\pi}/x^*).$$

(5.5) i.e. $E\,\gamma(y^*, \pi^*/x^*) < E\,\gamma(y^*, \bar{\pi}/x^*)$.

Further, since $(x^*, (y^*, \pi^*))$ satisfies the relation (5.2), i.e.

$$\theta(x^*, y^*, \pi^*) \geq \theta(x^*, y^*, \bar{\pi})$$

which implies that

(5.6) $$E\,\gamma(y^*, \pi^*/x^*) \geq E\,\gamma(y^*, \bar{\pi}/x^*).$$

We see that (5.6) contradicts the relation (5.5). Hence (y^*, π^*) must be optimal for the program (4.2). Theorem, is, thus, proved.

PART - II

Non-Linear Programming Under Risk and Uncertainty

1. **PROBLEM STATEMENT:**

The following programming problem is discussed in this section:

$$(1.1) \qquad \operatorname*{Inf}_{x \geq 0} \; E\left[g(c,x) + \operatorname*{Inf}_{y \geq 0}\left\{d'y\right\} + \operatorname*{Inf}_{z}\left\{h(z)\right\}\right]$$

subject to

$$(1.2) \qquad Ax \geq b$$

$$(1.3) \qquad P\left[Tx + My \geq p\right] \geq \alpha$$

$$(1.4) \qquad P\left[\phi(x) + \psi(z) \geq q\right] = 1$$

where g and h are convex scalar functions and each component of the vector valued functions ϕ and that of ψ are concave functions of their respective arguments. x,y and z are decision variables of order nx1, n_1x1 and n_2x1 respectively. A and M are mxn and m_1xn_1 fixed matrices, b and d are mx1 and n_1x1 fixed vectors c,p and q are ℓ x1, m_1x1 and m_2x1 random vectors and T is $m_1$xn random matrix. α is a probability vector each component of which lies between 0 and 1. 1 is an m_2x1 sum vector. P denotes the probability with respect to each component and E, the expectation. Prime denotes the transpose. All the random variables are assumed independent with known distribution functions.

This problem is known as the two-stage nonlinear programming problem under risk and uncertainty. The complementary probability $1 - \alpha_i$ represents the allowable risk such that the random variables will take on the values $T_i x + M_i y < p_i$, i=1,2,——m_1.

As in the Part I, it is assumed that the joint probability space $(\Omega, \mathcal{F}, P_r)$ is given, where Ω is a Borel subset of R^N, $N = \ell + m_1 n + m_1 + m_2$ and other terms as defined earlier. All other assumptions of the part I also hold here.

2. THE SOLUTION SET:

The decision vector for the problem (1.1) - (1.4) is called the 'here and now' decision as termed by Dantzig [34]. Once x is selected and the random variables are observed, the second stage decision variables can be obtained by solving the following problems:

(2.1) Inf $\{d'y\}$

(2.2) subject to $P\{M_i y \geqslant p_i - T_i x\} \geqslant \alpha_i$, $i=1,2,$-----m

(2.3) $y \geqslant 0$

and

(2.4) Inf $h(s)$

(2.5) subject to $\Psi(s) \geqslant q - \Phi(x)$.

The decision variable x may depend upon the second stage programming decisions and the corresponding costs obtained. Our main interest is to find out the decision x which is optimal for the program (1.1) - (1.4).

Let $K_1 = \{x/Ax \geqslant b, x \geqslant 0\}$

Then, K_1 is a convex polyhedron.

Let $K_{2q} = \{x/ \exists$ s such that $\Psi(s) \geqslant q - \Phi(x)\}$.

Then, for fixed q, the set K_{2q} is convex.

Let $K_2 = \bigcap_q K_{2q}$, assumed to be not null.

Then, K_2 is a convex set.

The constraints (1 3) can be written as follows:

(2.6) $\quad T_i x + M_i y \geqslant p_i$

(2.7) $\quad 1 - F_i(\eta_i) \geqslant \alpha_i$

where $\quad F_i(\eta_i) \equiv P\left[M_i y < p_i - T_i x = \eta_i\right]$

Let $\quad G_i(\eta_i) = 1 - F_i(\eta_i) \geqslant \alpha_i \geqslant 0.$

Then the derivative of

(2.8) $\quad G_i(\eta_i) = -f_i(\eta_i) \leqslant 0$

where $f_i(\eta_i)$ is a density function and $f_i(\eta_i) \geqslant 0.$

<u>Lemma 2.1</u>. If η_i has a continuous distribution function $G_i(\eta_i)$, then $G_i(\eta_i)$ is a quasi-monotonic function of η_i i e. both quasi-convex and quasi-concave function of η_i.

<u>Proof</u>. Let K_{η_i} be the domain of G_i such that $G_i(\eta_i) \geqslant 0$ for each i. Since $G_i(\eta_i)$ is continuous by hypothesis and is decreasing by (2.8), the set K_{η_i} is convex, and then, the lemma follows:

Now, let $K_{3Tp} = \{x / \exists\, y \geqslant 0$ such that $T_i x + M_i y \geqslant p_i,\ x \geqslant 0$ and $G_i(\eta_i) \geqslant \alpha_i,\ i=1,2,\text{----}m_1\}$

<u>Lemma 2.2</u>. For given T and p, the set K_{3Tp} is convex.

The proof is immediate if T and p are fixed, because then the constraints $G_i(\eta_i) \geqslant \alpha_i$, $i=1,2,\text{----}m_1$ and the decision vector y become irrelevant and the set K_{3Tp} becomes $\{x/Tx \geqslant p,\ x \geqslant 0\}$ which is obviously convex.

Also, if T and p are random, the set K_{3Tp} is convex.

Since, the set K_{η_i} is convex for given η_i, the set $\{x/G_i(p_i-T_ix) \geqslant \alpha$, $i=1,2,\text{------}m_1\}$ for given T and p, is convex. The set $\{x/ \exists\, y \geqslant 0$ such that $T_ix + M_iy \geqslant p_i$, $i=1,2,\text{------}m_1$, $x \geqslant 0\}$ is obviously convex for given T_i and p_i. Thus, the set $K_{3Tp} = \{x/G_i(\eta_{i\,1\,2\,m}) \geqslant \alpha\} \cap \{x/ \exists y \geqslant 0$ such that $T_ix + M_iy \geqslant p_i$, $x \geqslant 0$, $i=1,2,\text{------}m_1\}$ is convex for given T_i and p_i.

Let $K_3 = \bigcap\limits_{Tp} K_{3Tp}$ assumed to be not null

Then, the set K_3 is convex.

Consider, the set

$K = K_1 \cap K_2 \cap K_3$ assumed to be not null.

Then, the set K is convex. This set K is called the permanently feasible set of solutions.

3. THE EQUIVALENT CONVEX PROGRAM:

A decision equivalent deterministic program is obtained for the problem (1.1) - (1.4). The second stage program can be split up into two programs (2.1) - (2.3) and (2.4) - (2.5).

Let

(3.1) $R(x,q) = \{\text{Inf } h(s)/\ \psi(s) \geqslant q - \phi(x)\}$.

Then $R(x,q)$ is a convex function of x as is shown in the part I above.

For the first program (2.1)-(2.3) it is assumed that either the random variables T_i and p_i, $i=1,2,\text{------}m_1$ are normally distributed or they have known means and variances. Then, as seen in the previous chapter, (2.2) becomes

$$(3.2) \quad \overline{T_i}x - (x'V^i x)^{\frac{1}{2}} + M_i y \geq p_i^* \;, \; i=1,2,\text{------}m_1$$

where $\overline{T_i}x = E(T_i x) = E(T_i)x$, $V^i = t_i^2 B^i$, B^i is the variance and covariance matrix of T_i, $p_i^* = \overline{p}_i - t_i \sigma_{p_i}$ with $\overline{p}_i = E(p_i)$ and $\sigma_{p_i}^2 =$ the variance of p_i where t_i is determined by the probability law

$$P\left[\eta_i(x) + M_i y \geq \overline{\eta_i(x)} - t_i \sigma_{\eta_i(x)} + M_i y\right]$$

$$= P\left[\frac{\eta_i(x) - \overline{\eta_i(x)}}{\sigma \eta_i(x)} \geq -t_i\right] = \alpha_i, \; i=1,2,\text{------}m_1$$

where $\eta_i(x) = T_i x - p_i$, $\overline{\eta_i(x)} = E(\eta_i(x))$ and

$$\sigma_{\eta_i(x)}^2 = \begin{pmatrix} x \\ -1 \end{pmatrix}' \begin{bmatrix} B^i & 0 \\ 0 & \sigma_{p_i}^2 \end{bmatrix} \begin{pmatrix} x \\ -1 \end{pmatrix} = \text{variance of } \eta_i(x).$$

Here B^i is a positive-semi-definite matrix and σ_i, is a non-negative number, we have

$$\sigma_{\eta_i(x)} = \left[(x'B^i x) + \sigma_{p_i}^2\right]^{1/2} \geq (x'B^i x)^{\frac{1}{2}} - \sigma_{p_i}$$

which is used in obtaining (3.2).

Since the program (2.1) - (2.3) is linear in y,

let

$$\left\{\begin{matrix}\text{Inf} \\ y \geq 0\end{matrix} \quad d'y/\; M_i y \geq p_i^* - \overline{T_i}x + (x'V^i x)^{\frac{1}{2}}, \; i=1,2,\text{------}m_1 \right\}$$

$$= \sum_{i=1}^{m_1} \left[s_i p_i^* - s_i \overline{T_i}x + (x'Q^i x)^{\frac{1}{2}}\right]$$

where s_i's are some constants and $Q^i = s_i^2 V^i$, $i=1,2,\text{------}m_1$.

Let $E\,g(c,x)$ and $ER(x,q)$ exist and equal to $g(x)$ and $R(x)$ respectively. The functions $g(x)$ and $R(x)$ will be convex functions of x.

Also, $\sum_{i=1}^{m_1} \left[(x'Q^i x)^{\frac{1}{2}} - s_i \overline{T}_i x\right]$ is a convex function of x. The program (1.1) - (1.4), thus reduces to the following equivalent deterministic convex program

$$\text{(3.3)} \qquad \operatorname*{Inf}_{x \geq 0} \quad f(x) + c'x + \sum_{i=1}^{m_1} (x'Q^i x)^{\frac{1}{2}}$$

$$\text{(3.4)} \qquad \text{subject to} \quad Ax \geq b$$

where $f(x) = g(x) + R(x)$ and $c'x = \sum_{i=1}^{m_1} (- s_i \overline{T}_i x)$

and the constant term is dropped out. The objective function of (3.3) is convex and the set of feasible solutions is a convex polyhedron which is assumed to be bounded. By the property of the convex functions that they are bounded below over the bounded convex subsets of the set over which they are defined, the program (3.3) will obtain finite infimum. Thus the program (1.1) - (1.4) is proper i.e. will obtain finite value.

4. DUAL RELATIONS FOR THE EQUIVALENT CONVEX PROGRAM:

We consider the program (3.3) (3.4) with Inf. replaced by Min.

$$\text{Minimise } F(x) = f(x) + c'x + \sum_{i=1}^{m_1} (x'Q^i x)^{\frac{1}{2}}$$

$$\text{(4.1)} \qquad \text{subject to} \quad Ax \geq b$$

$$\text{(4.2)} \qquad x \geq 0$$

In this program, we assume that $f(x)$ is differentiable and bounded

The function $\sum_{i=1}^{m_1} (x'Q^i x)^{\frac{1}{2}}$ may not be differentiable somewhere in the domain of definition. Q^is are nxn non-negative definite symmetric matrices. Call this problem as primal denoted by (P). The dual of (P) is denoted by (D) and can be expressed as:

$$\text{Maximize} \quad G(v,u,u^1,\text{------}u^{m_1}) = f(u)-u'\nabla f(u) + b'v$$

(4.3) subject to $A'v - \nabla f(u) - \sum_{i=1}^{m_1} Q^i u^i \leq c$

(4.4) $u^{i'}Q^i u^i \leq 1, \ i=1,2,\text{------}m_1$

(4.5) $v \geqslant 0.$

Let C_P and C_D respectively denote the sets of feasible solutions for problems (P) and (D). It is assumed that the sets C_P and C_D are compact i.e. closed and bounded. Since $f(x)$ is differentiable and convex in x on C_P, we have

(4.6) $f(x) - f(u) \geqslant (x-u)'\nabla f(u)$

for $x,u \in C_P$

<u>Theorem 4.1</u>. For all feasible solutions to problems (P) and (D) respectively

(4.7) $\text{Sup } G(v,u,u^1,\text{------}u^{m}) \leq \text{Inf } F(x).$

<u>Proof</u>. Let $(v,v,u^1,\text{------}u^{m}) \in C_D$ and $x \in C_P$. Then,

$b'v \leq v'Ax$ (by (4.1) and (4.5))

(4.8) $\leq x'\nabla f(u) + \sum_{i=1}^{m_1} x'Q^i u^i + c'x$ (by (4.2) and (4.3))

$\leq f(x) - f(u) + u'\nabla f(u) + \sum_{i=1}^{m_1} (x'Q^i x)^{\frac{1}{2}} + c'x$

(by (4.6) and (4.2.16)).

Thus, by rearranging the terms in (4.8), we obtain

$$f(u) - u'\nabla f(u) + b'v \leq f(x) + \sum_{i}^{m_1} (x'Q^i x)^{\frac{1}{2}} + c'x$$

which implies that (4.7) is satisfied.

Hence, theorem is proved.

Let x^o be an optimal solution to the problem (P) Consider the following program denoted by (P_1)

$$\underset{x \in C_p}{\text{Minimise}} \quad F_1(x) = x'\nabla f(x^o) + c'x + \sum_{i=1}^{m_1} (x'Q^i x)^{\frac{1}{2}}$$

<u>Theorem 4.2.</u> A solution x^o to the problem (P) is optimal if and only if x^o is optimal for the problem (P_1).

<u>Proof</u>. Let x^o be an optimal solution to (P) and, if possible, let $x^* \in C_p$ such that

(4.9) $$F_1(x^*) < F_1(x^o).$$

Since C_p is convex, for all $\lambda \in (0,1)$ i.e. $0 < \lambda < 1$, $x_\lambda = \lambda x^* + (1-\lambda)x^o \in C_p$.

Then,

$$F(x_\lambda) = f\lfloor x^o + \lambda(x^* - x^o)\rfloor + \sum_{i=1}^{m_1} (x'_\lambda Q^i x_\lambda)^{\frac{1}{2}} + c'x_\lambda$$

$$= f(x^o) + \lambda(x^* - x^o)'\nabla f(x^o) + \frac{\lambda^2}{2}(x^* - x^o)' M(x^* - x^o)$$

$$+ \sum_{i=1}^{m_1} (x'_\lambda Q^i x_\lambda)^{\frac{1}{2}} + c'x_\lambda$$

(by the mean value theorem) where M is an nxn matrix whose ijth element is $\nabla_{x_i}\nabla_{x_j} f(x)$ for $i=1,2,\text{---}n$, $j=1,2,\text{---}n$.

$$\leq f(x^o) + \lambda(x^* - x^o)'\nabla f(x^o) + \frac{\lambda^2}{2}(x^* - x^o)' M(x^* - x^o)$$

$$+ \sum_{i=1}^{m1} (x^{o\prime} Q^i x^o)^{\frac{1}{2}} + \lambda \sum_{i=1}^{m_1} \left[(x^{*\prime} Q^i x^*)^{\frac{1}{2}} - (x^{o\prime} Q^i x^o)^{\frac{1}{2}}\right] + c'x_\lambda \quad .$$

Thus,

$$F(x_\lambda) - F(x^0) \leq \lambda (x^*-x^0)' \nabla f(x^0) + \frac{\lambda^2}{2} (x^*-x^0)' M(x^*-x^0) + \lambda c'(x^*-x^0)$$

(4.10) $$+ \lambda \sum_{i=1}^{m_1} \left[(x^{*'}Q^i x^*)^{\frac{1}{2}} - (x^{0'}Q^i x^0)^{\frac{1}{2}} \right]$$

$$= \lambda \left[F_1(x^*) - F_1(x^0) \right] + \frac{\lambda^2}{2} (x^*-x^0)' M(x^*-x^0)$$

The first term in the right-hand side of (4.10) is negative because of (4.9) and is independent of λ . It is possible to choose λ so small such that the right-hand-side of (4.10) remains negative i e.

$$F(x_\lambda) \leq F(x^0)$$

which contradicts the hypothesis that x^0 is optimal for (P).

Thus, the assumption (4.9) is invalid.

Conversely, if x^0 is an optimal solution to the problem (P_1) then,

(4.11) $\quad F_1(x^0) \leq F_1(x)$ for all $x \in C_P$.

Thus,

$$F(x^0) - F(x) = f(x^0) - f(x) + \sum_{i=1}^{m_1} \left[(x^{0'}Q^i x^0)^{\frac{1}{2}} - (x'Q^i x)^{\frac{1}{2}} \right]$$

$$+ c'(x^0-x)$$

$$\leq (x^0-x)' \nabla f(x^0) + \sum_{i=1}^{m_1} \left[(x^{0'}Q^i x^0)^{\frac{1}{2}} - (x'Q^i x)^{\frac{1}{2}} \right]$$

$$+ c'(x^0-x) \qquad \text{(by (4.6))}$$

$$= F_1(x^0) - F(x)$$

$$\leq 0 - \text{ (by (4.11))}.$$

which shows that x^0 is optimal for (P).

Hence, the Theorem is proved.

Theorem 4.3. If there exists a vector $x^o \in C_P$ minimizing $F(x)$ in C_P, then there exists $(v^o, u^o = x^o, u_o^1, \ldots, u_o^{m_1}) \in C_D$ maximizing $G(v, u, u^1, \ldots, u^{m_1})$ in C_D such that

$$F(x^o) = G(v^o, x^o, u_o^1, \ldots, u_o^{m_1}).$$

Proof. Let $x^o \in C_P$ be an optimal solution to the problem (P). Then, by Theorem 4.2, x^o is optimal for the problem (P_1). The dual to the problem (P_1) is denoted by (D_1) and expressed as:

$$\underset{v \geq 0}{\text{Max}} \left\{ b'v = G_1(v, x^o, u^1, \ldots, u^{m_1}) \right\}$$

$$\text{subject to} \quad A'v - \nabla f(x^o) - \sum_{i=1}^{m_1} Q^i u^i \leq c$$

$$u^{i'} Q^i u^i \leq 1, \; i = 1, 2, \ldots, m_1.$$

The set of feasible solutions to (D_1) is denoted by C_{D_1}. Then, $C_{D_1} \subset C_D$

Also, C_{D_1} is closed and bounded.

The program (P_1) is equivalent to the program

$$(P_2) \quad \underset{x \geq 0}{\text{Min}} \; F_1(x)$$

$$(4.12) \text{ subject to} \quad v'Ax \geq G_1(v, x^o, u^1, \ldots, u^{m_1})$$

in the sense that if x is feasible for (P_1), then for all $v \geq 0$,

$$v'Ax \geq b'v = G_1(v, x^o, u^1, \ldots, u^{m_1})$$

which implies that x is feasible for (P_2), and if x is feasible for (P_2), then

$$v'Ax \geq G_1(v, x^o, u^1 \ldots u^{m_1}) = b'v \text{ for all } v \geq 0$$

which implies that x is feasible for (P_1). Similarly with the help of [108, 48], (D_1) is equivalent to the problem

$$(D_2) \qquad \underset{v \geq 0}{\text{Max}} \; G_1(v, x^o, u^1, \text{------} u^{m_1})$$

$$\text{subject to} \quad v'Ax \leq F_1(x)$$

$$u^{i'} Q^i u^i \leq 1, \; i=1,2,\text{------} m_1$$

By the hypothesis x^o is optimal for (P_1), i.e. $x^o \in C_P \Rightarrow$ (by Theorem 4.4 below) that C_{D_1} is not null $\Rightarrow$ the dual problem (D_1) is feasible.

Since C_{D_1} is bounded, we have

$$A'v \leq 0, \; v \geq 0, \; G_1(v, x^o, u^1, \text{------} u^{m_1}) \geq 0 \implies v = 0.$$

Also, $F_1(x)$ and $G_1(v, x^o, u^1 \text{------} u^{m_1})$ are linear positively homogeneous convex and concave functions respectively Then, (by the theorem of duality in [49]) problems (P_2) and (D_2) have optimal solutions with their extreme values equal. Since x^o is optimal for (P_2), there exists $(v^o, x^o, u_o^1, \text{------} u_o^{m_1}) \in C_{D_1}$ such that

$$(4.13) \quad x^{o'} \nabla f(x^o) + \sum_{i=1}^{m_1} (x^{o'} Q^i x^o)^{\frac{1}{2}} + c'x^o = b'v^o.$$

Moreover, since $(v^o, x^o, u_o^1, \text{------} u_o^{m_1}) \in C_{D_1} \subset C_D \Rightarrow (v^o, x^o, u_o^1, \text{------} u_o^{m_1}) \in C_D$ i.e. it is feasible for (D). Now, we shall show that it is also optimal for (D).

For any other solution $(v, u, u^1, \text{------} u^{m_1}) \in C_D$, consider the following:

$$G(v^o, x^o, u_o^1, \text{------} u_o^{m_1}) - G(v, u, u^1, \text{------} u^{m_1})$$

$$= f(x^o) - f(u) + u' \nabla f(u) - x^{o'} \nabla f(x^o) + b'v^o - b'v$$

$$\geq (x^o - u)' \nabla f(u) + u' \nabla f(u) - x^{o'} \nabla f(x^o) + b'v^o - b'v \quad \text{(by (4.6))}$$

$$= x^{o'} \nabla f(u) + c'x^o + \sum_{i=1}^{m_1} (x^{o'} Q^i x^o)^{\frac{1}{2}} - b'v \quad (\text{by } (4.13))$$

$$\geqslant x^{o'} \nabla f(u) + \sum_{i=1}^{m_1} (x^{o'} Q^i u^i) + c'x^o - b'v$$

(by (4.2.16) and (4.4))

$$\geqslant v'Ax^o - b'v \quad (\text{by } (4.3) \text{ and } (4.2))$$

$$\geqslant 0 \quad (\text{by } (4.1) \text{ and } (4.5)),$$

which shows that $(v^o, x^o, u_o^1, \text{------} u_o^{m_1})$ is also optimal for (D). Also, by (4.13), we have

$$f(x^o) + \sum_{i=1}^{m_1} (x^{o'} Q^i x^o)^{\frac{1}{2}} + c'x^o = b'v^o + f(x^o) - x^{o'} \nabla f(x^o).$$

Thus, the theorem is proved.

<u>Theorem 4.4</u>. $C_P \neq \phi \Leftrightarrow C_{D_1} \neq \phi \Rightarrow C_D \neq \phi$,

where ϕ denotes the null set.

<u>Proof</u>. Let $C_P \neq \phi$. Then, $\exists\, x^o \in C_P$. Let if possible, $C_{D_1} = \phi$.

Then, $\nexists$ no solution to the system

$$A'v \leq \nabla f(x^o) + c, \; v \geqslant 0,$$

otherwise taking $u^i = 0$, $i=1,2,\text{------}m_1$, there will exist a solution belonging to C_{D_1}, thus contradicting the voidness of C_{D_1}. By the usual feasibility theorem [54,55], the following system

$$Ax \geqslant 0, \; (\nabla f(x^o) + c)'x < 0, \; x \geqslant 0$$

must have a feasible solution. This implies that for any $k \geqslant 0$, $x^o + kx \in C_P \Rightarrow C_P$ is unbounded, thus contradicting the hypothesis. Hence the assumption that $C_{D_1} = \phi$ is invalid. Similarly we can prove the converse part of this.

For the last part, we know that $C_{D_1} \subset C_D$, so $C_{D_1} \neq \phi \Rightarrow C_D \neq \phi$.

Corollary 4.1. $C_P = \phi \Leftrightarrow C_{D_1} = \phi \Leftarrow C_D = \phi$.

To establish the converse (duality Theorem) of Theorem 4.3, we reformulate the problem (D), in the following way as given in [38].

Let R_D denote the range of values of $u \in R^n$ such that (4 3) to (4.5) are satisfied. Let S denote the domain of the values of y in R^n such that the relation

$$A'v - y - \sum_{i=1}^{m_1} Q^i u^i \leq c$$

together with (4.4) and (4.5) is satisfied. Then, R_D is mapped onto S by

(4.14) $y = \nabla f(u)$

such that for a point $y \in S$, $\exists\, u \in R_D$ (not necessarily unique) satisfying (4.14). It is assumed that the function is once differentiable and moreover, in the neighbourhood of the maximum the mapping is one-one. It is assumed that S and R_D are convex.

Let $(v^o, y^o, u^o, u_o^1, \text{———} u_o^{m_1})$ be an optimal solution for (D) reformulated as follows

(D_3) Max $\psi(v, y, u, u^1, \text{———} u^{m_1}) = f(u) - u'y + b'v$

subject to

(4.15) $A'v - y - \sum_{i=1}^{m_1} Q^i u^i \leq c,$

(4.16) $u^{i\prime} Q^i u^i \leq 1,\ i=1,2, \text{———} m_1$

(4.17) $v \geq 0,$

(4.18) $y = \nabla f(u).$

The set of feasible solution to this problem is denoted by C_{D_3}.

Now consider the related linear problem denoted by (D_4)

$$\underset{v>0}{\text{Max}}\ \psi_1(v,y,u^0,u^1,\text{------}u^{m_1}) = -u^{0}y + b'v$$

$$\text{subject to}\quad A'v - y - \sum_{i=1}^{m_1} Q^i u^i \leq c$$

$$u^{i'}Q^iu^i \leq 1,\quad i=1,2,\text{------}m_1$$

<u>Theorem 4.5</u>. A solution $(v^0,y^0,u^0,u_0^1,\text{------}u_0^{m_1})$ is optimal for the problem (D_3) if and only if it is optimal for (D_4).

<u>Proof</u>. The first part of the proof is based on the similar lines as in [38].

For the proof of the other part, let $(v^0,y^0,u^0,u_0^1,\text{------}u_0^{m_1})$ be an optimal solution to (D_4). Then

(4.19) $\quad \psi_1(v^0,y^0,u^0,u_0^1\text{------}u_0^{m_1}) \geq \psi_1(v,y,u^0,u^1,\text{------}u^{m_1})$

for all $(u,y,u^0,u^1,\text{------}u^{m_1}) \in C_{D_4}$, the set of feasible solution to (D_4).

Since $y^0 \in S \Rightarrow \exists u^0 \in R_p$ such that $y^0 = \nabla f(u^0)$.

Similarly for $y \in S$, $\exists\, u \in R_p$ such that $y = \nabla f(u)$.

Then,

$$\psi(v^0,y^0,u^0,u_0^1,\text{------}u_0^{m_1}) - \psi(v,y,u,u^1,\text{------}u^{m_1})$$

$$= f(u^0) - f(u) - u^{0'}y^0 + u'y + b'v^0 - b'v$$

$$\geq (u^0-u)'\nabla f(u) - u^{0'}y^0 + u'y^0 + u'y + b'(v^0-v) \quad (\text{by } (4.6))$$

$$= \psi_1(v^0,y^0,u^0,u_0^1,\text{------}u_0^{m_1}) - \psi_1(v,y,u^0,u^1,\text{------}u^{m_1})$$

$$\geq 0 \quad (\text{by } (4.19)),$$

which implies that $(v^0,y^0,u^0,u_0^1,\text{------}u_0^{m_1})$ is optimal for (D_3).

Thus, the theorem is proved.

Theorem 4.6. (Converse of Theorem 4.3) If $(v^o, y^o, u^o, u_o^1, \cdots u_o^{m_1}) \in C_{D_3}$ is a maximising solution to (D_3), then $\exists\, x^o = u^o \in C_P$ minimizing $F(x)$ in C_P such that the extreme values of the two problems are equal.

Proof. Since $(v^o, y^o, u^o, u_o^1, \cdots u_o^{m_1}) \in C_{D_4}$ and is optimal for (D_4) by Theorem 4.5. The dual of (D_4) can be expressed as:

$$(P_3) \qquad \text{Min} \left\{ c'x + \sum_{i=1}^{m_1} (x'Q^ix)^{\frac{1}{2}} = F_2(x) \right\}$$

$$\text{subject to} \quad Ax \geqslant b, \quad x = u^o, \; x \geqslant 0,$$

and its set of feasible solutions is denoted by C_{P_3}. Then, it is clear that C_{P_3} is closed and bounded.

Writing $y = t - s$, $t \geqslant 0$, $s \geqslant 0$, the program (D_4) becomes

$$(D_5) \quad \text{Max} \; \psi_2(v,t,s,u^o,u^1,\cdots u^{m_1}) = \begin{pmatrix} b \\ -u^o \\ u^o \end{pmatrix}' \begin{pmatrix} v \\ t \\ s \end{pmatrix}$$

$$(4.20) \quad \text{subject to} \quad (A', -I, I) \begin{pmatrix} v \\ t \\ s \end{pmatrix} \leq c + \sum_{i=1}^{m_1} Q^iu^i$$

$$(4.21) \qquad u^{i'}Q^iu^i \leq 1, \; i=1,2,\cdots m_1$$

$$(4.22) \qquad v \geqslant 0, \; t \geqslant 0, \; s \geqslant 0.$$

By the similar arguments as employed in establishing the equivalence of the two programs (P_1) and (P_2) in Theorem 4.3, program (D_5) is equivalent to

$$(D_6) \quad \underset{v,t,s \geqslant 0}{\text{Max}} \; \psi_2(v,t,s,u^o,u^1,\cdots u^{m_1})$$

$$\text{subject to} \quad x'(A'-I,I) \begin{pmatrix} v \\ t \\ s \end{pmatrix} \leq F_2(x) \qquad \text{for all } x \geqslant 0 \text{ and}$$

$$u^{i'}Q^iu^i \leq 1, \; i=1,2,\cdots m_1.$$

The program (P_3) is equivalent to the program

(P_4)

$$\min_{x \geq 0} F_2(x)$$

$$\text{subject to } \begin{pmatrix} v \\ t \\ s \end{pmatrix}' (A', -I, I)'x \geq \psi_2(v,t,s,u^o,{}^1,\text{——}u^{m_1})$$

for all $v \geq 0$, $t \geq 0$, $s \geq 0$.

Since C_D is closed and bounded by assumption, the set C_{D_4} is closed and bounded. Thus, by Theorem 4.7 below, the set C_{P_3} is bounded and not null, i.e.

$$Ax \sim 0,\ F_2(x) \leq 0,\ x \geq 0 \Longrightarrow x = 0.$$

Moreover $F_2(x)$ and $\psi_2(v,t,s,u^o,u^1,\text{——}u^{m_1})$ are positively homogeneous convex and concave functions respectively Then, by the duality theorem in [49] the two problems (P_3) and (D_4) has their optimal solutions and their extrema are equal. That is, since $(v^o,t^o,s^o,u^o,u_o^1\text{—}u_o^{m_1})$ is optimal for (D_4), $\exists\ x^o = u^o$ optimal for (P_3) and such that

$$(4.23) \qquad c'u^o + \sum_{i=1}^{m_1} (u^{o'}Q^i u^o)^{\frac{1}{2}} = -u^{o'}y^o + b'v^o$$

$$\text{i.e.} \qquad c'u^o + \sum_{i=1}^{m_1} (u^{o'}Q^i u^o)^{\frac{1}{2}} = -u^{o'} \nabla f(u^o) + b'v^o$$

where $y^o = \nabla f(u^o)$. We see that $u^o \in C_{P_3} \subset C_P \Longrightarrow u^o$ is feasible for the problem (P). To show that it is also optimal for all $x \in C_P$ consider

$$F(u^o) - F(x) = f(u^o) - f(x) + \sum_{i=1}^{m_1} \left[(u^{o'}Q^i u^o)^{\frac{1}{2}} - (x'Q^i x)^{\frac{1}{2}}\right] + c'(u^o - x)$$

$$\leq (u^o - x)' \nabla f(u^o) - u^{o'} \nabla f(u^o) + b'v^o + b'v^o$$

$$- \sum_{i=1}^{m_1} x' Q^i u_o^i - c'x$$

(by (4.6), (4 23), (4.4) and (4.2.16))

$$= - x' \nabla f(u^o) - \sum_{i=1}^{m_1} x' Q^i u_o^i + b'v^o - c'x$$

$$= - x'A'v^o + c'x + b'v^o - c'x \text{ (by (4.3) and (4.2))}$$

$$0 \text{ (by (4.1) and (4.5))},$$

which shows that u^o is optimal for the problem (P). Moreover, by (4.23), we get

$$f(u^o) + \sum_{i=1}^{m_1} (u^{o\prime} Q^i u^o)^{\frac{1}{2}} + c'u^o = f(u^o) - u^{o\prime} \nabla f(u^o) + b'v^o.$$

Thus, the Theorem is proved.

<u>Theorem 4.7</u>. $C_{D_3} \neq \phi \Longleftrightarrow C_{P_3} \neq \phi \Rightarrow C_P \neq \phi$.

Proof ot this theorem goes exactly on the same lines as in the proof of Theorem 4.4.

<u>Corollary 4.2</u>. $C_{D_3} = \phi \Longleftrightarrow C_{P_3} = \phi \Leftarrow C_P = \phi$.

CHAPTER - VI

SYMMETRIC DUAL PROGRAMS WITH STANDARD ERRORS IN THE OBJECTIVE FUNCTIONS

INTRODUCTION:

It has been seen in the last two chapters that various types of stochastic programming problems converge to the similar type of equivalent deterministic programming problems. These programming problems come out to be non-differentiable convex programming problems. Some dual relations have been obtained for such programs in those chapters. In the present chapter, the extension of these programs to the cases of general symmetric dual and self-dual programs are considered. These problems are discussed in two different Parts. Part I consists of the study of the symmetric and self-duality for the quadratic type-programs with non-differentiable objective functions, while part II

consists of the discussion of the symmetric duality and self-duality for the general non-linear programs with differentiable plus non-differentiable objective functions. The non-differentiable terms in the objective functions of these problems are called standard error terms.

Symmetric duality and self-duality have been studied by various authors [33,39,40,61,82,87,88,89] for differentiable as well as non-differentiable programs. The results in this chapter are considered as generalisations of the results of Mehndiratta [88].

PART - I

QUADRATIC SYMMETRIC PROGRAMS WITH STANDARD ERRORS IN THE OBJECTIVE FUNCTIONS

1. PROBLEM STATEMENT:

The following naturally symmetric dual programs are considered:

Primal Problem

(P) Maximize $f(x,y) = h'x - \frac{1}{2} x'Sx - \frac{1}{2}y'Ty - \sum_{i=1}^{r} (x'Q^i x)^{\frac{1}{2}}$

(1.1) subject to $Ax \leq b + Ty + \sum_{j=1}^{t} P^j y_j$

(1.2) $y_j' P^j y_j \leq 1, \quad j=1,2,\text{---}t$

(1.3) $x \geqslant 0$

Dual Problem

(D) Minimize $g(w,u) = b'w + \frac{1}{2} w'Tw + \frac{1}{2} u'Su + \sum_{j=1}^{t} (w'P^j w)^{\frac{1}{2}}$

(1.4) subject to $A'w + Su + \sum_{i=1}^{r} Q^i u_i \geqslant h$

(1.5) $u_i' Q^i u_i \leq 1, \quad i=1,2,\text{---}r$

(1.6) $w \geqslant 0$

where A is m×n matrix, S, Q^i $(i=1,2,\text{----}r)$ and T, $P^j(j=1,2,\text{----}t)$ are respectively n×n and m×m non-negative definite symmetric matrices, 'h, x, u, u_i' $(i=1,2,\text{----}r)$ and 'b, y, w, y_j', $(j=1,2,\text{----}t)$ are n×1 and m×1 vectors respectively. Prime denotes the transpose.

The problems (P) and (D) form the special class of programming problems whose objective functions may not be differentiable at some points in the domain of consideration Thus, they belong to a class of non-differentiable programming problems.

2. ASSUMPTIONS AND NOTATION:

Let C_p denote the set of all $(x,y,y_1,\text{----}y_t)$ satisfying the constraints of the primal problem (P) and C_D, the set of all $(w,u,u_1,\text{----}u_r)$ satisfying the constraints of the dual problem (D). Both sets C_p and C_D are assumed to be closed and bounded.

The sets C_p and C_D are obviously convex. Every feasible solution for the problems (P) and (D) is an element of C_p or C_D. All the quantities considered are assumed to be reals. It is assumed that

(i) $\quad \sup f(x,y) = -\infty$ if $C_p = \phi$ and

(2.1)

(ii) $\quad \operatorname{Inf} g(w,u) = +\infty$ if $C_D = \phi$.

where ϕ denotes the voidset.

The following relations are used in the following section, the proofs for which can be seen in [37, 129a] .

(2.2) $\quad \frac{1}{2}\, x'Sx + \frac{1}{2}\, u'Su \geqslant u'Sx$

(2.3) $\quad \frac{1}{2}\, w'Tw + \frac{1}{2}\, y'Ty \geqslant y'Tw$

(2.4) $\quad \sum_{i=1}^{r} (x'Q^i u_i) \leq \sum_{i=1}^{r} (x'Q^i x)^{\frac{1}{2}} (u_i'Q^i u_i)^{\frac{1}{2}}$

and

$$(2.5) \qquad \sum_{j=1}^{t} (w'P^j y_j) \leq \sum_{j=1}^{t} (w'P^j w)^{\frac{1}{2}} (y_j' Q^j y_j)^{\frac{1}{2}}$$

where x, u, u_i $(i=1,2,\text{-----}r) \in R^n$, and w, y, y_j $(i=1,2,\text{----}m) \in R^m$.

3. DUALITY FOR THE SYMMETRIC PROGRAMS

Theorem 3.1. (Weak duality) Under the hypothesis (2.1),

$$\sup \; f(x,y) \leq \operatorname{Inf} \; g(w,u)$$

for all feasible solutions to problems (P) and (D) respectively.

Proof. Let $(x,y,y_1,\text{------}y_t) \in C_P$, i.e. feasible for (P) and

$$(w,u,u_1,\text{------}u_r) \in C_D, \text{ i e. feasible for (D)}$$

Then,

$$f(x,y) = h'x - \tfrac{1}{2}x'Sx - \tfrac{1}{2}y'Ty - \sum_{i=1}^{r} (x'Q^i x)^{\frac{1}{2}}$$

$$\leq h'x + \tfrac{1}{2}u'Su - u'Sx + \tfrac{1}{2}w'Tw - w'Ty$$

$$- \sum_{i=1}^{r} (x'Q^i x)^{\frac{1}{2}} (u_i' Q^i u_i)^{\frac{1}{2}}$$

(by (2.2), (2.3) and (1.5))

$$\leq x'A'w - w'Ty + \tfrac{1}{2}u'Su + \tfrac{1}{2}w'Tw$$

(by (2.4), (1.3) and (1.4))

$$\leq b'w + \sum_{j=1}^{t} w'P^j y_j + \tfrac{1}{2}u'Su + \tfrac{1}{2}w'Tw$$

(by (1.1) and (1.6))

$$\leq g(w,u) \qquad \text{(by (2.5) and (1.2))}$$

which implies that

$$\sup f(x,y) \leq \operatorname{Inf} (w,u).$$

<u>Theorem 3.2</u>. $C_p \neq \phi \Leftrightarrow C_D \neq \phi$.

<u>Proof</u>. Let $C_p \neq \phi$. Then, $\exists\,(x^o, y^o, y_1^o, \text{------} y_t^o) \in C_p$ Now, let, if possible, $C_D = \phi$. Then, $\exists$ no $(w, u, u_1, \text{------} u_r)$ such that

$$(3.1)\qquad A'w \geqslant h - Su \;,\quad w \geqslant 0$$

is satisfied, otherwise, by taking $u_i = 0$, $i=1,2,\text{------}r$, $(w,u,0,\text{------}0) \in C_D$, a contradiction to its voidness.

Let $u = p - q$, with $p \geqslant 0$, $q \geqslant 0$. Then, the system (3.1) can be expressed as follows:

$$(3.2)\qquad (A', S, -S)\begin{pmatrix} w \\ p \\ q \end{pmatrix} \geqslant h,\; w \geqslant 0,\; p \geqslant 0,\; q \geqslant 0 \;.$$

Since (3.1) has no solution, the system (3.2) has no solution.
By the usual feasibility theorem [54,55] , the following system

$$(3.3)\qquad Ax \leqslant 0,\; Sx = 0,\; x \geqslant 0,\; h'x > 0$$

must have a feasible solution. This, then, implies that for any

$k \geqslant 0$ $(x^o + kx, y^o, y_1^o, \text{-----} y_t^o) \in C_p \Rightarrow C_p$ is unbounded,

a contradiction to the boundedness of C_p. Hence, $C_D \neq \phi$.
The other part of the theorem can be established in the same way as above.

<u>Corollary 3.1</u>. $C_p = \phi \Longleftrightarrow C_D = \phi$.

Proof follows from the Theorem 3.2.

<u>Corollary 3.2</u>. $C_p \neq \phi$ and $C_D = \phi \Rightarrow \sup f(x,y) = +\infty$.

<u>Corollary 3.3</u>. $C_D \neq \phi$ and $C_p = \phi \Rightarrow \operatorname{Inf} g(w,u) = -\infty$.

Now, let $y = v - s$, $v \geqslant 0$, $s \geqslant 0$.

The problem (P), then, becomes

$$\max_{x,v,z \geq 0} (h',0,0)\begin{pmatrix}x\\v\\z\end{pmatrix} - \frac{1}{2}\begin{pmatrix}x\\v\\z\end{pmatrix}'\begin{bmatrix}S & 0 & 0\\0 & T & -T\\0 & -T & T\end{bmatrix}\begin{pmatrix}x\\v\\z\end{pmatrix} - \sum_{i=1}^{r}\left[\begin{pmatrix}x\\v\\z\end{pmatrix}'\begin{bmatrix}Q^i & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{pmatrix}x\\v\\z\end{pmatrix}\right]^{\frac{1}{2}}$$

(P_1) subject to (1.2) and

(3.4) $$(A, -T, T)\begin{pmatrix}x\\v\\z\end{pmatrix} \leq b + \sum_{j=1}^{t} P^j y_j$$

The corresponding set of feasible solutions is denoted by C_{P_1} which is also closed and bounded.

The dual problem of (P_1) is expressed as follows:

(D_1) $$\min_{w \geq 0} b'w + \frac{1}{2}\begin{pmatrix}u\\\beta\\\gamma\end{pmatrix}'\begin{bmatrix}S & 0 & 0\\0 & T & -T\\0 & -T & T\end{bmatrix}\begin{pmatrix}u\\\beta\\\gamma\end{pmatrix} + \sum_{j=1}^{t}(w'P^j w)^{\frac{1}{2}}$$

subject to (1.5) and

(3.5) $$(A, -T, T)'w + \begin{bmatrix}S & 0 & 0\\0 & T & -T\\0 & -T & T\end{bmatrix}\begin{pmatrix}u\\\beta\\\gamma\end{pmatrix} \geq \begin{pmatrix}h\\0\\0\end{pmatrix} - \sum_{i=1}^{r}\begin{bmatrix}Q^i & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{pmatrix}u_i\\\beta_i\\\gamma_i\end{pmatrix}$$

The constraints (3.5) are also expressed as follows:

(3.6) $$A'w + Su \geq h - \sum_{i=1}^{r} Q^i u_i$$

(3.7) $$-T\,w + T\beta - T\gamma \geq 0$$

(3.8) $$T\,w - T\beta + T\gamma \geq 0$$

From (3.7) and (3.8) we see that

(3.9) $$T\,w = T(\beta - \gamma) \quad \text{for all } w \;\; 0, \text{ and } \beta, \gamma \in R^m,$$

$$\Longrightarrow \quad w = \beta - \gamma .$$

Thus, by (3.9), the program (D_1) reduces to the program (D).
The set C_{D_1} of feasible solutions of the problem (D_1) is, then, identical with the set C_D.

Now, consider the following transformations

$$(3.10)\qquad \begin{pmatrix} h \\ 0 \\ 0 \end{pmatrix} = c, \quad \begin{pmatrix} x \\ v \\ z \end{pmatrix} = \eta, \quad (A, -T, T) = B,$$

$$\begin{bmatrix} S & 0 & 0 \\ 0 & T & -T \\ 0 & -T & T \end{bmatrix} = H, \text{ and } \begin{bmatrix} Q^i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = L^i, \; i=1,2,\text{----}r.$$

The problem (P_1) therefore can be written as follows

$$(P_2)\qquad \operatorname*{Max}_{\eta \geqslant 0} f(\eta) = c'\eta - \tfrac{1}{2}\eta' H\eta - \sum_{i=1}^{r} (\eta' L^i \eta)^{\frac{1}{2}}$$

subject to (1.2) and

$$(3.11)\qquad B\eta \leq b + \sum_{i=1}^{r} P^j y_j$$

The corresponding set of feasible solutions to (P_2) is C_{P_2} which is closed and bounded because of C_P and C_{P_1}.

Let $(\eta^0, y_1^0, \text{----} y_t^0) \in C_{P_2}$ such that it is optimal for the problem (P_2).

Then, consider the following program

$$(P_3)\qquad \operatorname*{Maximize}_{(\eta, y_1, \text{----} y_t) \in C_{P_2}} F(\eta) = c'\eta - \eta_0' H\eta - \sum_{i=1}^{r} (\eta' L^i \eta)^{\frac{1}{2}}$$

<u>Theorem 3.3</u>. The solution $(\eta^0, y_1^0, \cdots y_t^0) \in C_{P_2}$ is optimal for the problem (P_2) if and only if it is optimal for the problem (P_3).

The proof of this theorem is similar to the proof of the Theorem 4 2 in Chapter IV.

<u>Theorem 3.4</u>. If $(x^0, y^0, y_1^0, \cdots y_t^0) \in C_P$ is an optimal solution to the problem (P) i.e. if $(\eta^0, y_1^0, \cdots y_t^0) \in C_{P_2}$ is an optimal solution to the problem (P_2) then there exists $(w^0, \eta^0, \eta_1^0, \cdots \eta_r^0) \in C_{D_1}$ an optimal solution to the problem (D_1) i.e. $(w^0, u^0, u_1^0, \cdots u_r^0) \in C_D$, an optimal solution to the problem (D) with $x^0 = u^0$ and $y^0 = w^0$ such that the optimal values of the two programs are equal.

<u>Proof</u>. Let $\rfloor\, (x^0, y^0, y_1^0, \cdots y_t^0) \in C_P$ optimal for (P). Then $\exists$ a corresponding solution $(\eta^0, y_1^0, \cdots y_t^0) \in C_{P_2}$ optimal for (P_2). By Theorem 3.3, $(\eta^0, y_1^0, \cdots y_t^0)$ is also optimal for the program (P_3).

The dual of (P_3) can be expressed as

$$(D_2) \quad \underset{w \geqslant 0}{\text{Min}} \quad G(w, \eta^0, \eta_1, \cdots \eta_r) = b'w + \sum_{j=1}^{t} (w'P^j w)^{\frac{1}{2}}$$

$$(3.12) \text{ subject to } \quad B'w + H\eta^0 + \sum_{i=1}^{r} L^{i'} \eta_i \geqslant c$$

$$(3.13) \qquad \eta_i' L^i \eta_i \leq 1, \; i = 1, 2, \cdots r$$

The set of feasible solutions to (D_2) is denoted by C_{D_2}. It is clear that $C_{D_2} \subset C_{D_1}$.

By the similar arguments as employed in Theorem 4.5 Chapter IV, it can be easily shown that the program (P_3) is equivalent to the program

(P_4) $\quad \underset{\eta > 0}{\text{Max}} \quad F(\eta)$

subject to $\quad w'B\eta \leq G(w, \eta^o \eta_1, \text{------} \eta_r)$ for all $w \geqslant 0$.

Similarly, the program (D_2) is equivalent to the program

(D_3) $\quad \underset{w > 0}{\text{Min}} \quad G(w, \eta^o, \eta_1, \text{------} \eta_r)$

subject to $\quad \eta' B' w \geq F(\eta)$ for all $\eta \geqslant 0$.

Since, the solution to the problem (P_3) exists $\Longrightarrow$ that C_{P_2} is non-empty

$\Rightarrow$ (by Theorem 3.2) C_{D_2} is non-empty. Also, C_{D_2} is bounded, we have,

$$B'w > 0,\ w \geqslant 0,\ G(w, \eta^o, \eta_1, \text{------} \eta_r) \leq 0 \Rightarrow w = 0.$$

Furthermore, $G(w, \eta^o, \eta_1, \text{------} \eta_r)$ and $F(\eta)$ are continuous, positively homogeneous convex and concave functions respectively. Then, by the duality theorem for linear homogeneous programs [49], the two programs (P_4) and (D_2) have optimal solutions and their extreme values are equal. i.e. since $(\eta^o, y_1^o, \text{----} y_t^o) \in C_{P_3}$ is optimal for (P_4), there exists $(w^o, \eta^o, \eta_1^o, \text{----} \eta_r^o) \in C_{D_2}$ optimal for (D_2) such that

$$(3.14) \quad e'\eta^o - \eta^{o'}H\eta^o - \sum_{i=1}^{r} (\eta^{o'}L_i\eta^o)^{\frac{1}{2}} = b'w^o + \sum_{j=1}^{t} (w^{o'}F_j w^o)^{\frac{1}{2}}$$

Using the transformations (3.10), we obtain

$$(3.15) \quad h'x^o - \tfrac{1}{2}x^{o'}Sx^o - \tfrac{1}{2}y^{o'}Ty^o - \sum_{i=1}^{r}(x^{o'}Q_i x^o)^{\frac{1}{2}}$$

$$= b'w^o + \tfrac{1}{2}x^{o'}Sx^o + \tfrac{1}{2}y^{o'}Ty^o + \sum_{j=1}^{t}(w^{o'}F_j w^o)^{\frac{1}{2}}$$

From this and from (3.9), (3.12), (3.13), (1.6) we obtain that $(w^o = y^o, x^o, u_1^o, \text{------} u_r^o)$ is feasible for the problem (D). If $(w, u, u_1, \text{------} u_r)$ is any other feasible solution to (D), then, we have,

$$g(y^0,x^0,u_1^0,\text{———}u_r^0) - g\ (w,u,u_1,\text{———}u_r)$$

$$= b'y^0 + \tfrac{1}{2}\ x^{0'}Sx^0 + \tfrac{1}{2}y^{0'}Ty^0 + \sum_{j=1}^{t} (y^{0'}P^jy^0)^{\frac{1}{2}}$$

$$- b'w - \tfrac{1}{2}u'Su - \tfrac{1}{2}w'Tw - \sum_{j=1}^{t} (w'P^jw)^{\frac{1}{2}}$$

$$= h'x^0 - \tfrac{1}{2}\ x^{0'}Sx^0 - \tfrac{1}{2}y^{0'}Ty^0 - \sum_{i=1}^{r} (x^{0'}Q^ix^0)^{\frac{1}{2}}$$

$$- b'w - \tfrac{1}{2}\ u'Su - \tfrac{1}{2}\ w'Tw - \sum_{j=1}^{t} (w'P^jw)^{\frac{1}{2}} \quad (\text{by } (3.15))$$

$$\leq h'x^0 - x^{0'}Su - y^{0'}Tw - \sum_{i=1}^{r} (x^{0'}Q^iu_i) - \sum_{j=1}^{t} (w'P^jy_j^0) - b'w$$

(by (1.2), (1.5), (2.2), (2.3), (2.4) and (2.5))

$$\leq w'Rx^0 - y^{0'}Tw - \sum_{j=1}^{t} w'P^jy_j^0 - b'w$$

(by (1.4) and (1.3))

$$\leq 0 \quad (\text{by } (1.1) \text{ and } (1.6))$$

which implies that $(y^0,x^0,u_1^0,\text{———}u_r^0)$ is also optimal for the program (D). Thus, the Theorem is proved.

Theorem 3.5. (Converse of Theorem 3.4). If $(w^0,u^0,u_1^0,\text{———}u_r^0) \in C_D$ is an optimal solution to the problem (D), then there exists an optimal solution $(x^0,y^0,y_1^0,\text{———}y_t^0) \in C_P$ for the problem (P) with $y^0 = w^0$ and $x^0 = u^0$ and the extreme values of the two programs are equal.

Proof of this theorem can be easily established by the similar construction of the problem (D) as is done for the problem (P) in proving Theorem 3.4.

4. SELF-DUALITY

Theorem 4.1. The programming problem

$$(P_5) \quad \text{Maximize } \zeta(x,y) = h'x - \tfrac{1}{2}\, x'Sx - \tfrac{1}{2}\, y'Sy - \sum_{i=1}^{r} (x'Q^i x)^{\frac{1}{2}}$$

$$(4.1) \quad \text{subject to} \quad Dx + h \leq Sy + \sum_{i=1}^{r} Q^i y_i$$

$$(4.2) \qquad y_i' Q^i y_i \leq 1, \; i=1,2,\text{------}r$$

$$(4.3) \qquad x \geqslant 0$$

where D is nxn skew-symmetric and S and each of Q^i, i=1,2,------r, are nxn non-negative definite symmetric real matrices, is a self dual program. Moreover, if (P_5) is feasible program, then

$$\text{Max.} \quad \zeta(x,y) = 0$$

Proof. By relations (D) - (1.4) to (1.6) the dual of the program (P_5) can be written as

$$(P_6) \quad \text{Minimize} \quad \theta(w,u) = -h'w + \tfrac{1}{2}\, u'Su + \tfrac{1}{2}\, w'Sw + \sum_{i=1}^{r} (w'Q^i w)^{\frac{1}{2}}$$

subject to (1.5), (1.6) and

$$(4.4) \qquad D'w + Su + \sum_{i=1}^{r} Q^i u_i \geqslant h$$

Let C_{P_5} and C_{P_6} respectively denote the constraint sets for the problems (P_5) and (P_6). The set C_{P_5} is assumed to be closed and bounded. Since D is skew symmetric, we have $D = -D'$, and $x'Dx = 0$, for all $x \in R^n$, Then, the problem (P_6) can be rewritten as follows:

$$\text{Max:}\quad \theta(w,u) = h'w - \tfrac{1}{2}\, u'Su - \tfrac{1}{2}w'Sw - \sum_{i=1}^{r} (w'Q^{i}w)^{\frac{1}{2}}$$

(P_7) subject to (1.5), (1.6) and

(4.5) $$Dw = -D'w \leq -h + Su + \sum_{i=1}^{r} Q^{i}u_{i}$$

which is the problem (P_5) with $(x,y,y_1,\text{------}y_r)$ replaced by $(w,u,u_1,\text{-----}u_r)$.

This implies that (P_5) is a self-dual program and $C_{P_5} = C_{P_6}$.

Let $(x,y,y_1,\text{------}y_r) \in C_{P_5}$. Then, for $x \geqslant 0$

$$0 \geqslant x'Dx + h'x - x'Sy - \sum_{i=1}^{r} x'Q^{i}y_{i}$$

$$\geqslant h'x - \tfrac{1}{2}\, x'Sx - \tfrac{1}{2}\, y'Sy - \sum_{i=1}^{r} (x'Q^{i}x)^{\frac{1}{2}}$$

(4.6)

(by (2.2), (4.2), (2.4))

$$= \xi(x,y).$$

Thus, (4.6) implies that $\xi(x,y)$ is bounded above. Similarly it can be established that $\theta(w,u) \geqslant 0$, i.e. bounded below for $(w,u,u_1,\text{-------}u_r) \in C_{P_6}$

Now, let $(x^{o},y^{o},y_1^{o},\text{------}y_r^{o}) \in C_{P_5}$ be an optimal for (P_5). Then, by the duality Theorem 3.4 and Theorem 3.5, there exists $(x^{o}=w^{o},y^{o},u^{o},$ $y_1^{o}= u_1^{o},\text{-------}y_r^{o} = u_r^{o}) \in C_{P_6}$ which is optimal for (P_6) such that

$$\xi(x^{o},y^{o}) = \theta(w^{o},u^{o}).$$

But, since $\xi(x,y)$ and $\theta(w,u)$ are bounded above and bounded below respectively, we have

$$0 \geq \xi(x^0,y^0) = \theta(w^0,u^0) \geq 0$$

which implies that $\xi(x^0,y^0) = \theta(w^0,u^0) = 0.$

The theorem is, thus, proved.

PART - II

GENERAL NON-LINEAR SYMMETRIC PROGRAMS WITH STANDARD ERRORS IN THE OBJECTIVE FUNCTIONS

1. PROBLEM STATEMENT

In this section the following general non-linear symmetric programs are considered

Primal Program

(P) Minimize $F(x,y,y_1y_2\text{---}y_t) = f(x)+c'x+\sum_{i=1}^{r}(x'Q^ix)^{\frac{1}{2}}+y'\nabla g(y)-g(y)$

(1.1) subject to $Ax+\nabla g(y)+\sum_{j=1}^{t}F^jy_j \geq b$

(1.2) $y_j'F^jy_j \leq 1,\ j=1,2,\text{---}$

(1.3) $x \geq 0.$

Dual Program

(D) Maximize $G(v,u,u_1,\text{---}u_r)=f(u)-u'\nabla f(u)-g(v)-\sum_{j=1}^{t}(v'F^jv)^{\frac{1}{2}}+b'v$

(1.4) subject to $A'v \leq c+\nabla f(u)+\sum_{i=1}^{r}Q^iu_i$

(1.5) $u_i'Q^iu_i \leq 1,\ i=1,2,\text{---}r$

(1.6) $v \geq 0.$

In these problems $f(x)$ and $g(y)$ are assumed to be convex, bounded and differentiable scalar functions. Q^i, $i=1,2,\text{----}r$ and P^j, $j = 1,2,\text{----}t$ are nxn and mxm non-negative definite symmetric matrices respectively. A is mxn matrix, $x, c,$ and u_i, $i=1,2,\text{----}r$ are nx1 vectors and y, and y_j, $j=1,2,\text{----}t$ are mx1 vectors. Prime denotes the transpose. 0 is an appropriate dimensional vector. All the quantities considered belong to the reals.

Denote by C_P and C_D the sets of feasible solutions for the problems (P) and (D) respectively. It is assumed that C_P and C_D are closed and bounded. The convention similar to (2.1) of the previous section is also taken here.

2. DUALITY FOR THE GENERAL SYMMETRIC PROGRAMS

Theorem 2.1. For all feasible solutions to problems (P) and (D) respectively, we have

$$(2.1) \qquad \operatorname{Inf} \; F(x,y) \geqslant \sup G(v,u).$$

Proof. Let $(x,y,y_1,\text{----}y_t) \in C_P$ and $(v,u,u_1,\text{----}u_r) \in C_D$. Then, $(x,y,y_1,\text{----}y_t)$ and $(v,u,u_1,\text{----}u_r)$ are feasible for problems (P) and (D) respectively.

If C_P and C_D are empty sets, the relation (2.1) holds obviously by assumption.

Let $C_P \neq \phi$ and $C_D \neq \phi$, where ϕ denotes the null set.

Then

$$F(x,y,y_1,\text{----}y_t) \geqslant c'x + \sum_{i=1}^{r} (x'Q^i u_i) + f(u) - u'\nabla f(u) + x'\nabla f(u)$$
$$+ y'\nabla g(y) - g(v) + (v-y)\nabla g(y)$$

(by (1.5) and $(2.4)_I$, and by the convexity of $f(x)$ and $g(y)$)

$$\geqslant f(u) - u'\nabla f(u) - g(v) + v'Ax + v'\nabla g(y)$$

(by (1.3) and (1.4))

$$\geqslant b'v - \sum_{j=1}^{t} v'P^j y_j - g(v) + f(u) - u'\nabla f(u)$$

(by (1.1) and (1.6))

$$\geqslant b'v - \sum_{j=1}^{t} (v'P^j v)^{\frac{1}{2}} - g(v) + f(u) - u'\nabla f(u)$$

(by (1.2) and $(2.5)_I$)

$$= G(v, u, u_1 \text{———} u_r)$$

From this it is implied that (2.1) is satisfied.

Thus, the theorem is proved.

<u>Theorem 2.2.</u> If $(x^0, y^0, y_1^0 \text{———} y_t^0) \in C_P$ and $(v^0, u^0, u_1^0 \text{———} u_r^0) \in C_D$ and any feasible solutions for (P) and (D) respectively such that

(2.2) $\quad F(x^0, y^0, y_1^0, \text{———} y_t^0) = G(v^0, u^0, u_1^0 \text{———} u_r^0)$

then, $(x^0, y^0, y_1^0, \text{———} y_t^0)$ and $(v^0, u^0, u_1^0, \text{———} u_r^0)$ are optimal for their respective programs.

<u>Proof</u>. By Theorem 2.1, we have

$$G(v^0, u^0, u_1^0, \text{———} u_r^0) = F(x^0, y^0, y_1^0 \text{———} y_t^0) \geqslant G(v, u, u_1 \text{———} u_r)$$

for all $(v, u, u_1 \text{———} u_r) \in C_D$.

Similarly,

$$F(x, y, y_1 \text{———} y_t) \geqslant G(v^0, u^0, u_1^0 \text{———} u_r^0) = F(x^0, y^0, y_1^0 \text{———} y_t^0)$$

for all $(x, y, y_1, \text{———} y_t) \in C_P$.

Thus, the theorem is proved.

Under the similar construction as done in Chapter V part II,

Consider the following program denoted by (P_1)

(P_1) Minimize $H(x,z,y,y_1,\text{---}y_t) = c'x + \sum_{i=1}^{r} (x'Q^i x)^{\frac{1}{2}} + f(x) + y'z - g(y)$

subject to the constraints (1.2) and (1.3), and

(2.3) $Ax + z + \sum_{j=1}^{t} P^j y_j \geq b$

(2.4) $z = \nabla g(y)$

which is problem (P) rephrased in this form. The corresponding set of feasible solutions is denoted by C_{P_1}. Then, C_{P_1} is closed and bounded, since C_P is so.

Theorem 2.3. A solution $(x^0,z^0,y^0,y_1^0,\text{---}y_t^0) \in C_{P_1}$ is optimal for the problem (P_1) if and only if it is optimal for the problem.

(P_2) $\underset{x \geq 0, z \in S}{\text{Minimize}}$ $H_1(x,z,y_1,\text{---}y_t) = c'x + \sum_{i=1}^{r} (x'Q^i x)^{\frac{1}{2}} + x'\nabla f(x^0) + y^{0'}z$

subject to $Ax + z + \sum_{j=1}^{t} P^j y_j \geq b$, $y_j' P^j y_j \leq 1$, $j=1,2,\text{---}t$

where S is as defined in Chapter V.

The set of feasible solutions to the problem (P_2) is C_{P_2}.

Proof. Let $(x^0,z^0,y^0,y_1^0,y_t^0) \in C_{P_1}$ be an optimal for (P_1). Then $(x^0,z^0,y_1^0\text{---}y_t^0)$ is a feasible solution for the problem (P_2). Let, $(x^*,z^*,y_1^*,\text{---}y_t^*) \in C_{P_2}$ be another solution to (P_2) such that

(2.5) $H_1^* = H_1(x^*,z^*,y_1^*,\text{---}y_t^*) < H_1(x^0,z^0,y_1^0\text{---}y_t^0) = H_1^0$

Consider $x^\lambda = \lambda x^* + (1-\lambda)x^0$, $z^\lambda = \lambda z^* + (1-\lambda)z^0$, $y_1^\lambda = \lambda y_1^* + (1-\lambda)y_1^0$, $\text{---}, y_t^\lambda = \lambda y_t^* + (1-\lambda)\, y_1^0$, where $0 < \lambda < 1$.

Then, $(x^\lambda, z^\lambda, y_1^\lambda, \text{---}y_t^\lambda) \in C_{P_2}$ i.e. it is feasible for (P_2).

As $s^{\lambda} \in S$, there exists a corresponding point $y^{\lambda} \in R_D$ such that

(2.6) $$s^{\lambda} = \nabla g(y^{\lambda}).$$

Then, it follows that $(x^{\lambda}, s^{\lambda}, y^{\lambda}, y_1^{\lambda}, \text{———} y_t^{\lambda}) \in C_{P_1}$

Now, consider

$$\{ H^{\lambda} = H(x^{\lambda}, s^{\lambda}, y^{\lambda}, y_1^{\lambda} \text{———} y_t^{\lambda}) \} - \{ H(x^{0}, s^{0}, y^{0}, y_1^{0} \text{———} y_t^{0}) = H^{0} \}$$

$$= f(x^{\lambda}) + c'x^{\lambda} + \sum_{i=1}^{r} (x^{\lambda'} Q^{i} x^{\lambda})^{\frac{1}{2}} + y^{\lambda'} s^{\lambda} - g(y^{\lambda}) - f(x^{0}) - c'x^{0}$$

$$- \sum_{i=1}^{r} (x^{0'} Q^{i} x^{0})^{\frac{1}{2}} - y^{0'} s^{0} + g(y^{0})$$

$$= \lambda (x^{*} - x^{0})' \{ \nabla f(x^{0} - \theta(x^{\lambda} - x^{0})) - \nabla f(x^{0}) \} + \lambda (x^{*} - x^{0})' \nabla f(x^{0})$$

$$+ \lambda c'(x^{*} - x^{0}) + \sum_{i=1}^{r} \left[\lambda^{2} x^{*'} Q^{i} x^{*} + 2\lambda (1-\lambda) x^{*'} Q^{i} x^{0} + (1-\lambda)^{2} x^{0'} Q^{i} x^{0} \right]^{\frac{1}{2}}$$

$$- \sum_{i=1}^{r} (x^{0'} Q^{i} x^{0})^{\frac{1}{2}} - (y^{\lambda} - y^{0})' \{ \nabla g(y^{\lambda} - \beta(y^{\lambda} - y^{0})) - \nabla g(y^{\lambda}) \}$$

$$- (y^{\lambda} - y^{0})' \nabla g(y^{\lambda}) + y^{\lambda'} \nabla g(y^{\lambda}) - y^{0'} s^{0}$$

(by (2.6) and the mean value theorem, where $0 \leq \theta \leq 1$ and $0 \leq \beta \leq 1$)

$$\leq \lambda (x^{*} - x^{0})' \{ \nabla f(x^{0} - \lambda\theta(x^{*} - x^{0})) - \nabla f(x^{0}) \} + \lambda (x^{*} - x^{0})' \nabla f(x^{0})$$

$$+ \lambda c'(x^{*} - x^{0}) + \lambda \sum_{i=1}^{r} \left[(x^{*'} Q^{i} x^{*})^{\frac{1}{2}} - (x^{0'} Q^{i} x^{0})^{\frac{1}{2}} \right] \qquad + y^{0'}(s^{*} - s^{0})$$

$$- (s^{*} - s^{0})' H (s^{0} - \lambda\eta(s^{*} - s^{0})) \{ \nabla g(y^{\lambda} - \theta\lambda(y^{*} - y^{0})) - \nabla g(y^{\lambda}) \}$$

(by (2.2))

where $0 \leq \eta \leq 1$ and $H(s)$ is the square matrix whose ijth element is the partial derivative of the jth component of y with respect to the ith component of s.

Thus,

$$(2.7)\quad H^{\lambda} - H^{0} < \lambda\left[H_1^{*} - H_1^{0}\right] + (z^{*}-z^{0})'\, H\,(z^{0} - \lambda\eta(z^{*} - z^{0}))'\left\{\nabla g\,(y^{\lambda} - \theta\lambda(y^{*}-y^{0})) - \nabla g(y^{\lambda})\right\}$$

with $0 < \theta < 1$, and $0 < \eta \leq 1$.

The first term in the square bracket on the right-hand side of (2.7) is independent of λ, and since $\nabla g(y)$ and $\nabla f(x)$ are continuous, it is possible to make right-hand side negative by choosing λ arbitrarily small. Hence, $H^{\lambda} < H^{0}$, which is a contradiction to the hypothesis. This implies that $(x^{0}, z^{0}, y_1^{0}, \text{------} y_t^{0})$ is optimal solution of the problem (P_2).

Conversely, let $(x^{0}, z^{0}, y_1^{0} \text{------} y_t^{0}) \in C_{P_2}$ be an optimal solution for the problem (P_2). Since $z^{0} \in S$, by the assumption that the mapping is one-one at the optimal, there exists a point $y^{0} \in R_D$ such that $z^{0} = \nabla g(y^{0})$. This implies that $(x^{0}, z^{0}, y^{0}, y_1^{0} \text{------} y_t^{0})$ is a feasible solution to the problem (P_1).

Let $(x, z, y, y_1 \text{------} y_t) \in C_{P_1}$ be any other solution feasible for (P_1). We see that $(x, z, y_1, \text{------} y_t)$ is a feasible for problem (P_2).

Consider

$$H\,(x^{0}, z^{0}, y^{0}, y_1^{0}, \text{------} y_t^{0}) - H(x, z, y, y_1 \text{------} y_t)$$

$$= f(x^{0}) - f(x) + c'\,(x^{0}-x) + \sum_{i=1}^{r}\left[(x^{0\prime}Q^{i}x^{0})^{\frac{1}{2}} - (x'Q^{i}x)^{\frac{1}{2}}\right]$$

$$+\; y^{0\prime}z^{0} - y'z + g(y) - g(y^{0})$$

$$\leq (x^{0} - x)'\,\nabla f(x^{0}) + \sum_{i=1}^{r}\left[(x^{0\prime}Q^{i}x^{0})^{\frac{1}{2}} - (x'Q^{i}x)^{\frac{1}{2}}\right] + c'(x^{0}-x)$$

$$+ y^{o'}z^{o} - y'z + (y-y^{o})' \nabla g(y)$$

(by the convexity of f and g)

$$= H_1(x^o, z^o, y_1^o, \text{------} y_t^o) - H_1(x, z, y_1, \text{------} y_t) \leq 0.$$

Thus, $(x^o, z^o, y^o, y_1^o, \text{------} y_t^o)$ is also optimal for the problem (P_1).

Hence, the theorem is proved.

The problem (P_2) can be rephrased as follows:

(P_3) Minimise $H_2(x, \xi, \rho, y_1 \text{------} y_t)$

$$= \begin{pmatrix} c + f(x^o) \\ y^o \\ -y^o \end{pmatrix}' \begin{pmatrix} x \\ \xi \\ \rho \end{pmatrix} + \sum_{i=1}^{r} \left[\begin{pmatrix} x \\ \xi \\ \rho \end{pmatrix}' \begin{bmatrix} Q^i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ \xi \\ \rho \end{pmatrix} \right]^{1/2}$$

subject to

$$(2.8) \quad (A, I, -I) \begin{pmatrix} x \\ \xi \\ \rho \end{pmatrix} + \sum_{j=1}^{t} P^j y_j \geq b$$

$$(2.9) \quad y_j' P^j y_j \leq 1, \; j = 1, 2, \text{------} t,$$

$$(2.10) \quad x \geq 0, \; \xi \geq 0, \; \rho \geq 0.$$

The set of feasible solutions to this problem is denoted by C_{P_3}. Since, C_{P_2} is closed and bounded, C_{P_3} is so.

The dual of (P_3) can be expressed as follows:

$$(D_1) \quad \text{Maximise } G_1(v, x^o, u_1, \text{------} u_r) = b'v - \sum_{j=1}^{t} (v'P^j v)^{\frac{1}{2}}$$

$$(2.11) \quad \text{subject to } (A, I, -I)' v \leq \begin{pmatrix} c + f(x^o) + \sum_{i=1}^{r} Q^i u_i \\ y^o \\ -y^o \end{pmatrix}$$

and the problem

(D_2) $\quad \underset{v \geq 0}{\text{Maximise}} \quad G(v, x^0, u_1 \text{---} u_r)$

$$\text{subject to} \begin{pmatrix} x \\ \xi \\ \zeta \end{pmatrix} (A, I, -I)'v \leq H_2(x, \xi, \zeta, y_1, \text{---} y_t) \text{ for all}$$

$$x \geq 0, \; \xi \geq 0, \; \zeta \geq 0$$

Since, the problem (P_3) has an optimal solution $(x^0, \xi^0, \zeta^0, y_1^0, \text{---} y_t^0) \in C_{P_3}$ (by Theorem 2.4 above) that the problem (D_1) is feasible i.e. $C_{D_1} \neq \emptyset$. Since C_{D_1} is bounded, it implies that

$$A'v = 0, \; v \geq 0, \; G_1(v, x^0, u_1 \text{---} u_r) \geq 0 \implies v = 0 .$$

Furthermore, $H_2(x, \xi, \zeta, y_1, \text{---} y_t)$ and $G_1(v, x^0, u_1 \text{---} u_r)$ are continuous, positively homogeneous convex and concave functions respectively. By the duality Theorem of the homogeneous programming [49] the two problems (P_3) and (D_1) must have optimal solutions and their extreme values are equal.

Since $(x^0, \xi^0, \zeta^0, y_1^0, \text{---} y_t^0) \in C_{P_3}$ is optimal for (P_3), there exists $(v^0, x^0, u_1^0, \text{---} u_r^0) \in C_D$, optimal for (D_2) such that

$$(2.15) \quad c'x^0 + \sum_{i=1}^{r} (x^{0'} Q^i x^0)^{\frac{1}{2}} + x^{0'} \nabla f(x^0) + y^0(\xi^0 - \zeta^0) = b'v^0 - \sum_{j=1}^{t} (v^{0'} F^j v^0)^{\frac{1}{2}}$$

By (2.14), we see that $v^0 = y^0$.

Also, since $C_{D_1} \subset C_D$, $(y^0, x^0, u_1^0, \text{---} u_r^0) \in C_{D_1} \implies (y^0, x^0, u_1^0, \text{---} u_r^0) \in C_D$, i.e. it is feasible for the problem (D). To show that it is also optimal for (D), take $(v, u, u_1, \text{---} u_r)$ to be any other feasible solution for (D).

Then,

$$G(y^0, x^0, u_1^0, \text{———} u_r^0) - G(v, u, u_1, \text{———} u_r)$$

$$= c'x^0 + \sum_{i=1}^{r} (x^{0'} Q^i x^0)^{\frac{1}{2}} + x^{0'} \nabla f(x^0) + y^{0'} \nabla g(y^0) - g(y^0)$$

$$+ f(x^0) - x^{0'} \nabla f(x^0) - b'v + g(v) + \sum_{j=1}^{t} (v' P^j v)^{\frac{1}{2}} - f(u) + u' \nabla f(u)$$

(by (2.15) and since $\xi^0 - \zeta^0 = z^0 = \nabla g(y^0)$)

$$\geq c'x^0 + \sum_{i=1}^{r} (x^{0'} Q^i u_i) + y^{0'} \nabla g(y^0) - g(y^0) + f(x^0) - b'v + g(v)$$

$$+ \sum_{j=1}^{t} v' P^j y_j^0 - f(u) + u' \nabla f(u)$$

(by (1.2), (1.5) and (2.4)$_I$, (2.5)$_I$)

$$\geq c'x^0 + \sum_{i=1}^{r} (x^{0'} Q^i u_i) + (v - y^0)' \nabla g(y^0) + (x^0 - u)' \nabla f(u)$$

$$+ y^{0'} \nabla g(y^0) - b'v + \sum_{j=1}^{t} v' P^j y_j^0 + u' \nabla f(u)$$

(by the convexity of f and g)

$$\geq v'Ax^0 + v' \nabla g(y^0) - b'v + \sum_{j=1}^{t} v' P^j y_j^0$$

(by (1.4) and since $x^0 \geq 0$)

≥ 0 (by (1.1) and (1.6))

which implies that $(v^0, y^0, x^0, u_1^0, \text{———} u_r^0)$ is optimal for the problem (D). Also, by (2.15) we have

$$f(x^0) + c'x^0 + \sum_{i=1}^{r} (x^{0\prime} Q^i x^0)^{\frac{1}{2}} + y^{0\prime} \nabla g(y^0) - g(y^0)$$

$$= \quad = b'y^0 - \sum_{j=1}^{t} (y^{0\prime} P^j y^0)^{\frac{1}{2}} + f(x^0) - x^{0\prime} \nabla f(x^0) - g(y^0),$$

The converse can be proved by the similar arguments as above.

Theorem 2.6. If both problems (P) and (D) are feasible, then both problems have optimal feasible solutions and their optimal values are equal.

The proof is immediate after the application of Theorems 2.3, 2.4, and 2.5.

3. SELF-DUAL PROGRAMS

Let $h(x,y) = f(x) - g(y)$. Again, let $\nabla_1 h(x,y)$ and $\nabla_2 h(x,y)$ be the vector valued functions which are gradients of $h(x,y)$ with respect to x and y respectively (the first and the second argument of h respectively)

The symmetric programs (P) and (D), then, can be rewritten as follows:

(P_3) Minimize $H(x,y,y_1,\cdots,y_t) = c'x + \sum_{i=1}^{r} (x'Q^i x)^{\frac{1}{2}} + h(x,y) - y' \nabla_2 h(x,y)$

subject to $Ax - \nabla_2 h(x,y) + \sum_{j=1}^{t} P^j y_j \geq b$

$y_j' P^j y_j \leq 1, \; j=1,2,\cdots,t$

$x \geq 0$

and

(D_3) Maximize $G(v,u,u_1,\cdots,u_r) = b'v - \sum_{j=1}^{t} (v'P^j v)^{\frac{1}{2}} + h(u,v) - u' \nabla_1 h(u,v)$

subject to $A'v \leq c + \nabla_1 h(u,v) + \sum_{i=1}^{r} Q^i u_i$

$u_i' Q^i u_i \leq 1, \; i=1,2,\cdots,r$

$v \geq 0.$

The function $h(x,y)$ is said to be symmetric if $h(x,y) = h(y,x)$, and skew-symmetric if $h(x,y) = -h(y,x)$ for all (x,y) in the domain of h with x and y having the same dimension.

<u>Theorem 3.1</u>. The following problem

(P_6) Minimize $\quad H(x,y,u_1,\text{------}u_r) = c'x + \sum_{i=1}^{r} (x'Q^i x)^{\frac{1}{2}} + h(x,y) - y'\nabla_2 h(x,y)$

(3.1) subject to $\quad Ax - \nabla_2 h(x,y) + \sum_{i=1}^{r} Q^i u_i \geqslant -c$

(3.2) $\quad u_i'Q^i u_i \leq 1, \; i=1,2,\text{------}r$

(3.3) $\quad x \geqslant 0$

where $h(x,y) = f(x) - f(y)$ is skew symmetric differentiable function, A is skew-symmetric $n\times n$ matrix and other terms as defined earlier, is a self-dual program. Moreover, if (P_6) is feasible, the

$$\text{Min. } H(x,y,u_1,\text{------}u_r) = 0.$$

<u>Proof</u>. Similar to the relation (D_3) above, the dual of the problem (P_6) can be written as

(P_7) Max $\quad G(v,u,u_1,\text{------}u_r) = -c'x - \sum_{i=1}^{r} (v'Q^i v)^{\frac{1}{2}} + h(u,v) - u'\nabla_1 h(u,v)$

$$\text{subject to} \quad A'v \leq c + \nabla_1 h(u,v) + \sum_{i=1}^{r} Q^i u_i$$

$$u_i' Q^i u_i \leq 1, \; i=1,2,\text{------}r$$

$$v \geqslant 0.$$

Since h is a skew-symmetric function and A is a skew-symmetric matrix, we have

$$\nabla_1 h(u,v) = -\nabla_2 h(v,u)$$

and $\quad A = -A', \quad x'Ax = 0.$

Then, the program (P_7) can be rewritten as follows:

$$(P_8) \quad \text{Minimize} \quad c'v + \sum_{i=1}^{r} (v'Q^i v)^{\frac{1}{2}} + h(v,u) - u' \nabla_2 h(v,u)$$

$$\text{subject to } Av + \sum_{i=1}^{r} Q^i v_i - \nabla_2 h(v,u) \geqslant -c$$

$$u_i'Q^i u_i \leq 1, \; i=1,2,\text{----}r$$

$$v \geqslant 0$$

which is the problem (P_6) with x and y replaced by v and u respectively. That is, if $(x^o,y^o,u_1^o,\text{----}u_r^o)$ is an optimal feasible for the problem (P_6), then, $(v^o=x^o,u^o=y^o,u_1^o,\text{-----}u_r^o)$ is optimal feasible for the problem (P_8). Hence, the problem (P_6) is a self-dual program.

Now, let $(x,y,u_1,\text{----}u_r) \in C_{P_6}$ = the set of feasible solutions for the problem (P_6). Then,

$$0 \leq x'Ax + \sum_{i=1}^{r} (x'Q^i u_i) - x' \nabla_2 h(x,y) + c'x$$

(by (3.3) and (3.1))

$$\sum_{i=1}^{r} (x'Q^i x)^{\frac{1}{2}} + c'x + h(x,y) - y' \nabla_2 h(x,y)$$

(by (3.2), $(2.4)_I$ and by the convexity of f)

$$\leq \quad H(x,y,u_1,\text{-----}u_r).$$

This implies that $H(x,y,u_1,\text{-----}u_r)$ is bounded below. Similarly, for all $(v,u,u_1,\text{-----}u_r) \in C_{P_6}$, it can be shown that $G(v,u,u_1,\text{-----}u_r)$ is bounded above, i.e. $G(v,u,u_1\text{-----}u_r) \leq 0$. Let $(x^o,y^o,u^o_1,\text{-----}u_r^o) \in C_{P_6}$ be an optimal solution for the problem (P_6). Then, by the duality theorem 2.5

above, there exists $(v^0=x^0, u^0=y^0, u_1^0, \text{---} u_r^0)$ optimal for the problem (P_7) and such that

$$H(x^0, y^0, u_1^0, \text{---} u_r^0) = G(v^0, u^0, u_1^0, \text{---} u_r^0).$$

Since, the function H is bounded below by zero and the function G is bounded above by zero, we get

$$0 \leq H(x^0, y^0, u_1^0, \text{---} u_r^0) = G(v^0, u^0, u_1^0, \text{---} u_r^0) \leq 0$$

which implies that $H(x^0, y^0, u_1^0, \text{---} u_r^0) = G(v^0, u^0, u_1^0, \text{---} u_r^0) = 0.$

Thus, the theorem is proved.

BIBLIOGRAPHY

1. Arbusova, N.I. and Danilov, V.L. : "A Problem on Stochastic Linear Programming and its Stability" Dokl. Acad. Nauk. SSSR. Vol. 162 (1965).

2. Babbar, M.M. : "Distributions of Solutions of a set of Linear Equations" J. Amer. Statist. Assoc. 50, 1955.

3. Beale, E.M.L. : "On Minimizing a Convex Function Subject to Linear Inequalities" J. Royal Statist. Soc. Ser. B. 17, 1955.

4. Brems, J.P. : "Programmes Lineares Stochastiques" Cahiers Cent. Etud. Rech. Oper. Vol. 6, 1964.

5. Bui, Trong Lieu : "On a problem of Convexity and its Applications to Nonlinear Stochastic Programming" J. Math Anal. Appl. Vol. 8, 1964.

6. ______________ : "A Study of Some Inequalities for Nonlinear Stochastic Programming" In 'Nonlinear Programming' edr. J. Abadie, North Holland Publ. Col. 1967.

7. Bereanu, B. : "On Stochastic Linear Programming I. Distribution Problems: A Single Random Variable" Rev. Math. Pure Appl. VIII, 1963.

8. ______________ : "Stochastic Transportation Problem I. Case of Random Costs" Com. Acad. R.P. Romaine 13(1963).

9. ______________ : "Solutions of Minimum Risk in Linear Programming" An. Univ. Bucuresti. Sti. Natur. Mat. Mec. 13(1964).

10. ______________ : "Regions de Decision et Repartion de l'optimum dans la Programmation Lineare" CR Acad. Sci. Paris, 259 (1964).

Bereanu, B. : "On Stochastic Linear Programming II Distribution Problems Non-stochastical Technological Matrix" Rev. Roum. Math Pure Appl. Vol. 11 (1966)

2. ________ : "On Stochastic Linear Programming the Laplace Transform of the Distribution of the Optimum and Applications" J. Math. Anal. Appl. Vol. 15, 1966.

3. ________ : "On Stochastic Linear Programming:Distribution Problems. Stochastic Technological Matrix" Z Wahr. Verw Geb. 8, 1967.

4. Bracken, J. and Soland, R.M. : "Statistical Decision Analysis of Stochastic Linear Programming Problems" Nav. Res. Log. Qrly. Vol. 13, No. 3, 1966.

14a). Boot, J.C.G. : "Quadratic Programming" North Holland Publ. Co. Amsterdam, 1964.

15) Charnes, A. and Cooper, W.W. & Sy-monds, G.H. : "Cost Horizons and Certainty Equivalents: An Approach to Stochastic Programming of Heating Oil Production" Man. Sci. Vol. 4, No. 3, 1958.

6. ________ and ________ : "Chance-Constrained Programming" Man. Sci. Vol. 6, No. 1, 1959.

7. ________ and ________ : "Chance-Constrained Programs with Normal Deviates and Linear Decision Rules" Nav Res. Log Qrly 7, 1960

8. ________ and ________ : "Chance-Constraints and Normal Deviates" J. Amer. Statist. Assoc. 57, 1962.

9. ________ and ________ : "Deterministic Equivalents for Optimising and Satisficing under Chance-Constraints" Oprs. Res. Vol. 11, 1963.

20. ________ and ________ & Thompson, G.L. : "Characterisation by Chance-Constrained Programming" in Recent Advances in Mathematical Programming" Graves/Wolfe. edrs. 1963.

21. ________ and ________ & ________ : "Critical Path Analysis Via Chance-Constrained and Stochastic Programming" Oprs. Res. Vol.12, 1964.

22. ________ and ________ & ________ : "Chance-Constrained Programming and Related Approaches to Cost Effectiveness" Man. Sci. Res. Rep. 39, Carn. Inst. Tech. Rittsburgh, Pennsylvania, 1965.

23. Charnes, A. and Cooper, W.W. & Thompson, G.L. : "Constrained Generalized Medians and Hypermedians as Deterministic Equivalents for two Stage Linear Programs under Uncertainty" Man. Sci. 12, 1965.

24. ___________ and Dreze : "Decision Horizon Rules for Statistical Planning Problems. A Linear Example" Econometrica, 34, 2, 1966.

25. ___________ and Stedry, A.C. : "Chance-Constrained Model for Real Time Control in Research and Development Management" Man. Sci. 12, 1966.

26. ___________ and Kirby, M : "Optimal Decision Rules for the E-Model of Chance-Constrained Programming" Cah. Cent. Etud. Rech. Oper. Jan. 1966.

27. ___________ and ___ & Raike, W.M. "Chance-Constrained Generalized Networks" Opre. Res. 14, 1966.

28. ___________ and ___ & ___________ "Solution Theorems in Probabilistic Programming: A Linear Programming Approach." J. Math. Anal. Appl. 20, 1967.

29. ___________ and ___ & ___________ "Optimal Rejection Regions for a class of Probabilistic Programming Problems" SRM-182. North Western Univ. Evanston Illinois, 1967.

30. ___________ and ___ & Littlechild, S.C. and Raike, W.M. "Chance-Constrained Models for Pricing and Scheduling under Competition" SRM-183, North West Univ. Evanston, Ill., 1967.

31. ___________ and ___ & "Some Special P. Models in Chance-Constrained Programming" Man. Sci. 14, 1967.

32. ___________ and ___ & Raike, W.M. "Chance-Constrained Games with Partially Controllable Strategies" Opre. Res. 16, 1968.

33. Cottle, R.W. : "Symmetric Dual Quadratic Programs" Quart. Appl. Math. 21, 1963.

34. Dantzig, G.B. : "Linear Programming Under Uncertainty" Man. Sci. Vol. 1, No. 324, 1955.

35. ___________ and Madansky, A. : "On the Solution of Two-Stage Linear Programs under Uncertainty" Proc. IV Berkeley Symp. on Math. Statist. and Probab. Vol. I, Univ. of Calif., Berkeley, 1961.

36 Dempster, M.A.H. : "On Stochastic Programming I. Static Linear Programming Under Risk" J. Math. Anal. Appl. 21, 1968.

37. Dorn, W.S. : "Duality in Quadratic Programming" Qrly. Appl. Math XVIII, 1960.

38. __________ : "A Dualtiy Theorem for Convex Programs" IBMJ Research and Development. Vol. 4, 1960.

39. __________ : "Self - Duality in Quadratic Programming Problems" J. Soc. Indust Appl. Math. Vol. 9, 1961.

40. Dantzig, G.B. ; Eisenberg, E and Cottle, R.W. : "Symmetric dual Nonlinear Programs" ORC Rep. 30, Univ. of Calif. Berkeley, 1962.

41. Dunford and Schwarz : "Linear OperatOns Part I " Interscience Public. Co. Inc New York, 1964.

42. Elmaghraby, S.E. : "An Approach to Linear Programming Under Uncertainty" Oprs. Res. 7, 1959.

43. __________ : "Allocation under Uncertainty when Demand has Continuous D.F." Man. Sci. Vol. 6, 1960.

44. El-Agizy, M : "Programming under Uncertainty with Discrete D.F." ORC-64-13 (RR) July 1964, Univ. of Calif. Berkeley.

45. __________ : "Two-Stage Programming Under Uncertainty with Discrete Distributions Functions" Oprs. Res. Vol. 15, 1967.

46. __________ : "Dynamic Inventory Models and Stochastic Programming" Mobil Oil Corpn. N.Y. 1967.

47. Evers, W.H. : "A New Model for Stochastic Linear Programming" Man. Sci. Vol. 13, 1967.

48. Eisenberg, E. : "Supports of a Convex Function" Bull. Amer. Math. Soc. Vol. 68, 1962.

49. __________ : "Duality in Homogeneous Programming" Proc. Amer. Math. Soc. Vol. 12, 1961.

50. Ferguson, A.R. and Dantzig, G.B. : "The Allocation of Aircraft to Routes: An Example of Linear Programming Under Uncertainty" Man. Sci. Vol. 2, 1956.

51. Fruend, R.J. : "The Introduction of Risk into a Programming Model" Econometrica, 24, 1956.

52. Fisher, C.S. : "Linear Programming Under Uncertainty in an L Space" Tech. Rep. No 7, Dec. 1962, Univ. of Calif., Berkeley

53. Falk, J E. : "Lagrange Multipliers and Nonlinear Programming" J. Math. Anal Appl 19, 1967.

54. Fenchel, W : "Convex, Cones, Sets and Functions" Princeton Univ Lecture-notes, Princeton, N.J. 1953.

55 Gale, D. : "The Theory of Linear Economic Models" McGraw Hill Ney York, 1960.

56. Garvin, W.W. : "Introduction to Linear Programming" McGraw Hill, New York, 1960.

57. Geoffrion, A.M. : "A Prametric Programming Solution to the Vector-Maximum Problem with Applications to decision under Uncertainty" Tech Rep. 11, Grad. School of Business, Stanford Univ., 1965.

58 ____________ : "Stochastic Programming with Aspiration or Fractile Criteria" Man. Sci. Vol. 13, No. 9, 1967.

59. Hanson, M.A. : "Errors and Stochastic Variations in Linear Programming" The Austral J Statist. Vol. 2, 1960.

60. Hanson, M.A. : "Stochastic Nonlinear Programming" J. Austral. Math. Soc. Vol. 4, 1963.

61. ____________ : "Duality and Self-Duality in Mathematical Programming" J. Soc. Indust Appl. Math. Vol. 12, 1964.

62. Hadley, G. : "Nonlinear and Dynamic Programming" Addison Wesley 1964.

63. Hillier, F.S. : "Chance-Constrained Programming with 0-1 or Bounded Continuous Decision Variables" Man. Sc. Vol. 14, 1967.

64. Hurwicz, Leonid : "Programming in Linear Spaces" In Arrow K.J. etal. "Studies in Linear and Nonlinear Programming" Stanford Univ. Press, 1958.

65. Halmos, P.R : "Measure Theory" Van Nostrand, New York, 1950.

66. Huard, P. : "Dual Programs" in "Recent Advances in Mathematical Programming" Edr. Graves/ Wolfe, 1963, McGraw Hill N.Y

67. Iosifiscus, M. and Theodrorscu, R : "Statistical Decisions and Linear Programming" CR Acad. Bulgare Sci: 17(1964)

68. Kataoka, Shinji : "On Stochastic Programming I Stochastic Programming and its Applications to Production Horizon Problems II of Preliminary Study of Stochastic Model III, A Stochastic Programming Model. IV A Note on Generalized Stochastic Programming Model" Hitotsubashi J. Arts. Sci. Vol. 2 and 3, 1962 and 1963.

69. ______________ : "A Stochastic Programming Model" Econometrica, 31, 1963.

70. King, W.R. : "A Stochastic Assignment Model" Oprs. Res. 13, 1965.

71. Kirby, M.J.L. : "The Current State of Chance-Constrained Programming" SRM-9, North West Univ. Evanston, Ill. 1967.

72. Kortanek, K O. and Soden, J V. : "On The Chance-Kirby Optimality Theorem for the Conditional Chance-Constrained E-model" Cah.-Cent. Etud Rech. Opem. Vol. 9, 1967.

73. Kall Peter : "Des Zweistufige Problem der Stochastiche Lineare Programming" Z. Wahr. & Verw. Geb. 8, 1967.

73a. Kunzi, H.P. : "Nonlinear Programming" Blaisdell Publ. Comp. London, 1966.

74. Madansky, A. : "Bounds on the Expectation of a Convex Function of a Multi-variate Random Variable" Ann. Math. Statist. Vol. 30, 1959

75. ______________ : "Inequalities for Stochastic Linear Programming Problems" Man. Sci. Vol. 6, 1960.

76. ______________ : "Methods of Solutions of Linear Programs under Uncertainty" Rand. Rep. P-2132, 1960, Oprs. Res. 10, 1962.

77. ______________ : "Dual Variables in Two-Stage Linear Programming Under Uncertainty" J. Math. Anal. Appl. Vol. 6, 1963.

78. Madansky, A. : "Linear Programming Under Uncertainty" in Graves/Wolfe's "Recent Advances in Mathematical Programming" McGraw Hill, 1963.

79. Mangasarian, O.L : "Nonlinear Programming Problems with Stochastic Objective Functions" Man Sci. Vol. 10, 1964.

80. ______ and Rosen, J.B. : "Inequalities for Stochastic Nonlinear Programming Problems" Vol. 12, 1964, Oprs Res

81. ______ : "Duality in Nonlinear Programming" Quart. Appl. Math Vol. XX, 1962.

82. ______ and Ponstein, J. : "Minimax and Duality in Nonlinear Programming" J, Math. Anal. Appl Vol 11, 1965.

83. Manne, A.S. : "Linear Programming and Sequential Decisions" Man. Sci Vol. 6, 1960.

84. Miller, B.L. and Wagner, H.M. : "Chance-Constrained Programming with Joint Constraints" Oprs. Res. Vol. 13, 1965.

85. Murty, K.G. : "Two-Stage Linear Programs Under Uncertainty: A Basic Property of the Optimal Solution" ORC-66-4, Univ. of Calif., Berkeley, 1966.

86. Martos, Bela, : "The Direct Power of Adjacent Vertex Methods" Man. Sci. 1966

87. Mehndiratta, S.L : "General Symmetric Dual Programs" Oprs. Res. Vol. 14, 1966.

88. ______ : "Symmetry and Self-Duality in Nonlinear Programming" Numerische Mathematique. Vol. 10, 1967.

89. Mond, B. and Cottle, R.W. : "Self-Duality in Mathematical Programming" J. SIAM Appl. Math. Vol. 14, 1966.

90. Naslund, A. and Whinston, A.W. : "A Model of Multiperiod Investment under Uncertainty" Man. Sci. Vol. 8, 1962.

91. Prekopa, A. : "On the Problem Distribution of the Optimum of a Random Linear Program" SIAM J. Control, Vol. 4, 1966.

92. Reiter, S. : "Surrogates for Uncertain Decision Problems. Minimal Information for Decision making" Econometrica, 25, 1957.

93. Radner, R : "The Application of Linear Programming to Team Decision Problems" Man. Sci. Vol. 5, 1959.

94. Reisman, A. ; Rosenstein, A.B. & Buffa, E.S. : "Resource Allocation under Uncertainty and Demand Interdependence" J. Indust. Engg. Vol. 17, 1966.

95. Rutenberg, D.P. : "Linear Programming under Uncertainty with Discrete Probability Distributions" Working Paper No 166, Univ. of Calif. Berkeley, 1966

96. Raike, W.M. : "Chance-Constrained Games and Generalized Networks" Ph.D. Thesis, submitted to North West. Univ. Ill. 1967

97. ______________ : "Dissection Methods for Solutions in Chance-Constrained Programming Problems under Discrete Distributions" series in Appl. Math. for Management Publ. No AMM-2, The Univ. of Texas at Austin, 1968.

98. ______________ : "Applications of Rejection Region Theory to the Solution and the Analysis of Two-Stage Chance-Constrained Programming Problems" Series in App. Math Mang Publ. No. AMM-3, The Univ. of Texas at Austin, 1968.

98a. Rao, C.R. : "Linear Statistical Inference and its Applications" John Wiley, N.Y 1965.

99. Simon, H.A. : "Dynamic Programming under Uncertainty with Quadratic Criterion Function" Econometrica, Vol. 24, 1956.

100. Shetty, C.M. : "A Solution to Transportation Problem with Non-Linear Costs" Oprs. Res. Vol. 7, 1959.

101. Sengupta, J.K. ; Tintner, G and Millham, C. : "On Some Theorems of Stochastic Linear Programming with Applications" Man. Sci. Vol. 10, 1963.

102. ______________ & Morrison, B. : "Stochastic Linear Programming with Applications to Economic Models" Economica, 30, 1963.

103. ______________ & Kumar T K : "An Application of Sensitivity Analysis to a Linear Programming Problem" Unternehmensforschung" Vol. 9, 1965.

104. ______________ & Millham, C. and Tintner, G. : "On the Stability of Solutions under Error in Stochastic Linear Programming" Metrika, Vol. 9, 1965.

05. Sengupta, J.K. : "The Stability of Truncated Solutions of Stochastic Linear Programming" Econometrica, 34, 1966.

06. Sinha, S.M. : "Stochastic Programming" Rep. 63-22(RR) ORC. Univ. of Calif., Berkeley, 1963.

07 ________ : "A Duality Theorem for the Nonlinear Programming" Man. Sci. Vol. 12, 1966.

08 ________ : "An Extension of a Theorem on Supports of a Convex Function" Man. Sci. Vol. 12, 1966.

09. Szware, W. : "The Transportation Problem with Stochastic Demand" Man. Sci. Vol. 11, 1964.

10. Stan Fromovitz : "Nonlinear Programming with Randomization" Man. Sci. Vol. 11, 1965.

11. Soldatov, V.E : "On a Problem of Linear Programming with Random Restrictions" Sibirsk Mat Z. Vol. 6(1965)

12. Symonds, G.H. : "Chance-Constrained Equivalents of Some Stochastic Programming Problems" ORC. Univ. of Calif. Berkeley, 1967.

13. ________ : "Deterministic Solutions for a Class of Chance-Constrained Programming Problems" Oprs. Res. 15, 1967.

14. Sachan, R.S. : "Some Basic Inequalities in Stochastic Nonlinear Programming Problems" IIT/K. Math. Res. Rep. 1967, IIT Kanpur.

15. ________ : "Duality in Quadratic Programming Problems Under Uncertainty" Sent for Publication (1967).

16. ________ : "Two-Stage Nonlinear Programming under Uncertainty" Sent for Publication (1967-68).

17. ________ : "On the Nonlinear Stochastic Programming Problems" Cah. Cent. Etud. Rech. Oper. Vol. 10, 1968 pp. 84-99.

18. ________ : "Stochastic Programming Problems under risk and Uncertainty" Accepted in Cah. Cent. Etud. Rech. Oper. Vol. 12, 1970.

119. Sachan, R.S. : "On the Nonlinear Programming Problems under Risk and Uncertainty" To appear in the Bulletin of Operations Research Society of India.

120. ______________ : "Symmetric and Self-dual Program with Standard Errors in the Objective Functions" Sent for Publication (1968).

121. ______________ : "General Symmetric and Self-dual Programs with Standard Errors in the Objective Functions" Sent for Publication (1968).

122. Simmons, G.F. : "Introduction to Topology and Modern Analysis" McGraw Hill, N.Y. 1963

123. Theil, H. : "A note on certainty Equivalence in Dynamic Planning" Econometrica, Vol. 25, 1957.

124. Tintner, G. : "Stochastic Linear Programming with Applications to Agricultural Economics" Proc. II Symp. on Linear Programming National Bureau of Standards, D.C. 1955.

125. ______________ : "A Note on Stochastic Linear Programming" Econometrica 28, 1960.

126. ______________ : "The use of Stochastic Linear Programming in Planning" The Indian Economic Review, Vol. 5, 1960.

127. ______________ & Millham, C. and Rao, V.Y. : "An Application of Stochastic Linear Programming to Development Planning" Metro Economica, Vol. 14, 1962.

128. ______ & ______ & Sengupta, J.K. : "Weak Duality for Stochastic Linear Programming" Unternehmensf. Vol. 7, 1963.

129. ______ & Sengupta, J.K. : "On the Stability of Solutions under Recursive Programming" Unternah. Vol. 10, 1966.

129a. Taylor, A.E. : "Introduction to Functional Analysis" John Wiley & Sons. London, 1958.

130. Vajda, S. : "Inequalities in Stochastic Linear Programming" Bull. Intern. Statist. Inst. Vol. 36, 1958.

131. ______________ : "Mathematical Programming" Addison Wesley, London, 1961.

132. Votaw, D.F. Jr : "Statistical Programming" Ann. Math. Statist. Vol. 31, 1960.

133. Van de Panne, C. & Popp, W. : "Minimum Cost Cattle Feed under Probabilistic Protein Constraints" Man. Sci. Vol. 9, 1963.

134. ______________ : "Optimal Strategy Decisions for Dynamic Linear Decision Rules in Feed form". Econometrica 33, 1965.

135 Van Moeseke, P. : "Truncated Minimax Maximax Approach to Linear Programming under Risk" Econometrica 31, 1963.

136. ______________ : "Dynamic Risk Programming with Learning Adjustments" Unterneh. Vol. 7, 1963.

137. ______________ & Tintner, G. : "Base Duality Theorem for Stochastic and Parametric Linear Programming" Unterneh. Vol. 8, 1964.

138. ______________ : "Stochastic Linear Programming: A Study in Resource Allocation Under Risk" Yale Economic Essays, Vol. 5, 1965.

139. Van Slyke, R. & Wets, R. : "Programming under Uncertainty and Stochastic Optimal Control" J. SIAM Control, Vol. 4, 1966.

140. __________&______ : "L-Shaped Linear Programs with Applications to Optimal Control and Stochastic Programming" Boeing Document D1-82-05-26, 1966.

141. Wagner, H.M. : "On the Distribution of Solutions in Linear Programming Problems" J. Amer. Statist. Assoc. 53, 1958.

142. Williams, A.C. : "A Stochastic Transportation Problem" Oprs. Res. Vol. 11, No. 5, 1963.

143. ______________ : "On Stochastic Linear Programming" J. SIAM Appl. Math. Vol. 13, 1965.

144. ______________ : "Approximation Formulas for Stochastic Linear Programming J. SIAM Appl. Math. Vol. 14, 1966.

145. Wilson, R. : "On Programming Under Uncertainty" Oprs. Res. Vol. 14, No. 4, 1966.

146. Wets, R.J.B. : "Programming Under Uncertainty: The Equivalent Convex Program" J. SIAM Appl. Math. Vol. 14, 1966.

147. ______________ : "Programming Under Uncertainty: The Solution Set." J. SIAM Appl. Math. 14, 1966.

148. Wets, R.J.B. : "Programming Under Uncertainty: The Complete Problem" Z. Wahr. Verw. Geb. Vol. 4, 1966.

149. Walkup, D.W. & Wets, R.J.B. : "Stochastic Programs with Recourse" J. SIAM Appl. Math. Vol. 15, 1967.

150. ______ & ______ : "Stochastic Programs with Recourse: Special Forms" Boeing Document, Aug. 1967.

151. Weil, Jr. R.L. : "Functional Solution for the Stochastic Assignment Model" Oprs. Res. Vol. 15, 1967.

152. Williams, A.C. & Avriel, M. : "Remarks on Linear Programming under Uncertainty" Oprs. Res. Vol. 16, 1968.

153. Wolfe, P. : " A Duality Theorem for Nonlinear Programming Qrly. Appl. Math. Vol. XIX, No. 3, 1961.

154. Zitka Zaskova : "Stochastic Linear Programming" (Summary) Ekonomicko-Matematicky. Obzor 3, 1967,

Date Due

MATH-1969- D - SAE SOM.